Self-destructive percolation, invasion percolation and related models

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Thomas Stieltjes Institute
for Mathematics

VRIJE UNIVERSITEIT

Self-destructive percolation, invasion percolation and related models

ACADEMISCH PROEFSCHRIFT

ter verkrijging van de graad Doctor aan
de Vrije Universiteit Amsterdam,
op gezag van de rector magnificus
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ten overstaan van de promotiecommissie
van de faculteit der Exacte Wetenschappen
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door

Bálint Vágvölgyi

geboren te Salgótarján, Hongarije

promotoren: prof.dr. R.W.J. Meester
prof.dr. J. van den Berg

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Chapter 1

Introduction

1.1 Introduction

This thesis concerns itself with three different spatial stochastic models that have two important features in common. First, they are all "percolation-like" models. Second, they are connected to some extent to self-organized criticality in the sense that they either exhibit self-organizing behaviour or are closely related to models that do. This introductory chapter will give an overview of the concept of the self-organized criticality and introduce the models discussed in later chapters. Since a proper understanding of Bernoulli percolation is of essence when reading this thesis, a detailed overview of that model is also given. The main aim of this chapter is to give a clear intuitive picture of the considered model and motivate the choice of studying them. Therefore, while proper mathematical rigour is maintained throughout this chapter, the governing tone will be kept somewhat informal and focus on helping to gain insight of the models and the problems studied in this thesis. Therefore, non-mathematician readers are also encouraged to read this chapter.

1.1.1 Criticality and self-organized criticality

In a classical mechanics formulation systems in the universe are governed by the laws of nature. Therefore, if we have complete knowledge of the state of a system, the future influences and the laws of nature, we then can perfectly predict the complete future of the system. However, usually we do not possess such knowledge. In such situations it is often appropriate to describe the system using stochastic models to account for the uncertainties caused by our insufficient knowledge. Other systems, such as quantum system, have an intrinsic randomness, making a probabilistic interpretation a necessity.

Furthermore, macroscopic systems are made up of small particles that have insignificant size compared to the total size of the system and also individually insignificant influence to its global state. Such particles can for instance be atoms, molecules or even trees of large forests. The basic idea of modeling such systems is to describe the local interactions between particles but then study the global behaviour of the system. Due to our incomplete knowledge the microscopic interactions are often modeled by stochastic rules. Surprisingly, this microscopic randomness often does not transfer to the macroscopic level. Consider a toss of a fair coin. If the coin is tossed only once the outcome is perfectly unpredictable. However, if we toss it a thousand times we can predict the fraction of heads with remarkable accuracy. Therefore, we can say that the number of heads in one toss is completely random while in thousand tosses it is almost deterministic. The mathematical reason behind this is well understood and called the law of large numbers. Other situations are less clear but often the high number of the microscopic particles gives rise to non-trivial but deterministic qualitative properties on the global level. One may recall the famous quote by Heraclitus: “You could not step twice into the same river”. On one hand, Heraclitus was indeed right since the water flows constantly and the local structure of the river is ever changing. On the other hand however, anyone who stepped into the same river twice can tell that the look and feel of the river remain the same. Assuming of course that no serious pollution took place between the two occasions in which case the two experiences may differ significantly.

Often the stochastic models used to describe the microscopic behaviour depend on some parameter. As the parameter varies sometimes drastic changes in the global behaviour of the system are observed. This phenomenon is called phase transition. One of the most natural example of the phase transition is the water turning from ice to liquid when the temperature rises. in case of models that exhibit phase transition the parameter space can be divided into disjoint subsets. Within each set the main qualitative properties of the system are the same or at least very similar. Significant change in the state takes place when the parameter moves into another subset, i.e. when phase transition happens. Even though models may very well have more parameters, typically we talk about phase transition in one parameter. The models that are considered in this thesis have only one phase transition but it is possible to have two or more. A good example again is the water that has two phase transitions and therefore, three significantly different states.

It is very interesting to study the properties of a model with the parameter value set to be on the boundary of two phases. If the system and, in particular, the phase transition satisfies further conditions we call these states critical. The most appealing feature of the critical models is the scale-free behaviour. That is if we look at the system through a window, the observed picture does not depend on the size of the window as long as we allow for a certain minimal scale. In mathematical language such behaviour can be expressed with power law equations and fractal-like shapes. Power law behaviour mean that the probability distribution of the main quantities of the system obey a power law with a certain exponent and a fractal is an object that has roughly the same structure on every scale. Power laws are a typical characteristics of critical models while, naturally, fractal structures arise in spatial models. All the models considered in this thesis show both characteristics.

The need to study critical models naturally arises as many real world phenomena have scale-invariant behaviour. However, in classical models of spatial stochastics and statistical physics scale-invariance is observed only in the critical state and one should not expect nature to fine-tune the parameters of natural models to their critical values, especially if those models are dynamic. This paradox gives rise to studying a new class of dynamic models that drive themselves into a critical state without the need of tuning any parameter. This concept is called self-organized criticality and was first introduced by Bak, Tang and Wiesenfeld in [6]. A good general introduction to self-organized criticality can be found in Bak's book [5]. Self-organized critical (SOC) models studied by mathematicians include the Abelian sandpile model, forest fires and models for biological evolution, such as the Bak-Sneppen model.

Carrying out mathematically rigorous analysis of SOC models is typically difficult. Although physicists have predicted several power law exponents in different models, even ordinary critical models posed great difficulties for mathematicians. Until recently, only a few power law exponents were known rigorously and even those mostly due to some special feature of their respective models and not to some unified methodology for proving existence of such exponents. We will say more on this subject later in this chapter.

1.1.2 Bernoulli percolation

The percolation model was introduced by Broadbent and Hammersley in [14]. Broadbent, who was a mining engineer, approached Hammersley with the following problem concerning design of gas masks: "Gas molecules, absorbed on the surface of a porous medium move by surface diffusion through all pores large enough to admit them. What proportion of the interior of the solid will be reached by the gas and how does it depend on the porosity of the medium?"

The following model, proposed to study this problem, has later become well-known as Bernoulli or independent bond percolation. Consider the graph induced by $\mathbb{Z}^2$. That is the vertices (mostly referred to as sites in percolation theory) are the point of $\mathbb{Z}^2$ and there is an edge between two sites if their Euclidian distance is 1. Fix a $p \in [0, 1]$ and declare each edge open with probability p and closed with probability $1 - p$ independently of each other. The probability measure associated with this model is a product measure and is denoted by $\mathbb{P}_p$. This procedure results in a random subgraph of $\mathbb{Z}^2$ that turns out to have significantly different structure for different values of p. In the problem formulated for the gas mask the underlying graph, $\mathbb{Z}^2$, corresponds to the porous solid and gas can travel through open edges only. The value of p "measures" the porosity of the medium. Percolation-like models can be very broadly applied, for instance for modeling ferro-magnetism, social, electrical and computer networks, forest fires, spreads of epidemics etc. There are also

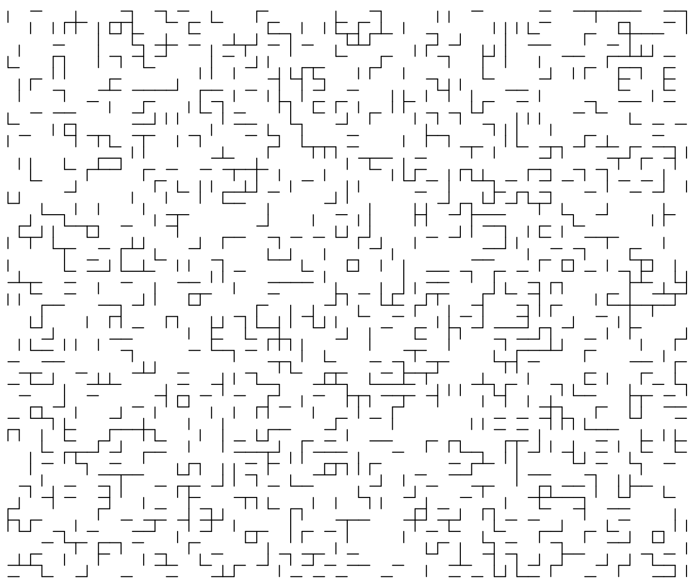

Figure 1.1: A realization of Bernoulli bond percolation with parameter $p = 0.25$ on a 60 by 50 window of the square lattice.

many natural modifications and generalizations of the above described percolation model. The most immediate is the independent site percolation. In this case all the bonds are open in the random configuration but each *site* can be open or closed with probability p and $1-p$, respectively. Another possible extension is considering different lattices. The triangular or honeycomb lattice is another commonly used planar lattice but percolation has also been extensively studied on $\mathbb{Z}^d$ for $d > 2$ and on trees. One can also relax the independence assumption and introduce dependencies between the states of, typically, neighbouring bonds. A well-known example of dependent percolation models is the random-cluster model by Fortuin and Kasteleyn; see [23]. Finally, one can define continuum percolation models without underlying graphs such as the Voronoi percolation (see [12] for a general introduction), the confetti percolation (see Problem 5 in [8]) or the Poisson Boolean models (an overview of these models can be found in [38]). We do not go into details of how these models are defined.

In this thesis we only consider planar lattices in fact our models will be defined on $\mathbb{Z}^2$. It is important to mention, though, that virtually all our results hold also for the triangular lattice. Now, recalling the problem of the gas masks, it is clear that the degree of penetration by the gas depends on the size of the open clusters. For a given bond e of $\mathbb{Z}^2$, the open cluster of e is the maximal connected set of open bonds that contain e. The behaviour of the model is clearly dependent on p. If, for instance, $p = 0.25$ as on Figure 1.1, then a typical configuration of open and closed edges will contain small separated open components surrounded by large closed clusters.

Due to obvious symmetries we observe just the opposite picture on Fig-

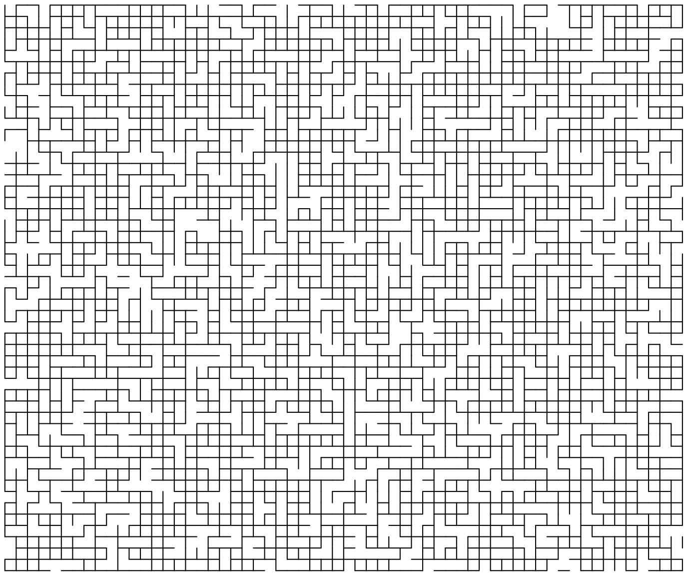

Figure 1.2: A realization of Bernoulli bond percolation with parameter $p = 0.75$ on a 60 by 50 window of the square lattice.

ure 1.2, when $p = 0.75$. In this case the open bonds form a large connected cluster interspersed with small closed islands.

However, the system appears to behave differently if $p = 0.5$, see Figure 1.2, when there are both large open and closed components. In fact it turns out that they occur on every scale and the $p = 0.5$ value corresponds to the critical state.

In order to formulate the concept of criticality we introduce the percolation function. If x and y are two distinct sites in $\mathbb{Z}^2$, the event $\{x \leftrightarrow y\}$ means that there exists a sequence of distinct sites $\{v_i\}_{i=1}^n$ such that $v_1 = x$, $v_n = y$ and for all j the vertices v_j and v_{j+1} are connected by an open bond. If $\{x \leftrightarrow y\}$ holds, we say that x and y are connected by an open path. The event $\{x \leftrightarrow \infty\}$ is defined analogously. It holds if there exists an infinite sequence of distinct vertices $\{v_i\}_{i=1}^\infty$ such that $v_1 = x$ and for all j the vertices v_j and v_{j+1} are connected by an open bond. The percolation function is defined as

$$\Theta(p) = \mathbb{P}_p(0 \leftrightarrow \infty). \tag{1.1}$$

Note that the graph is translation invariant and therefore, 0 plays no special role in the definition of $\Theta(p)$ and could be replaced by an arbitrary vertex of $\mathbb{Z}^2$. Clearly $\Theta(0) = 0$, $\Theta(1) = 1$ and $\Theta(p)$ is an increasing function of p. Therefore, we can define the critical probability p_c as

$$p_c = \inf\{p : \Theta(p) > 0\}.$$

Note that the definition of p_c also depends on the underlying lattice. For the sake of simplicity we omit this from our notations. As the name suggests,

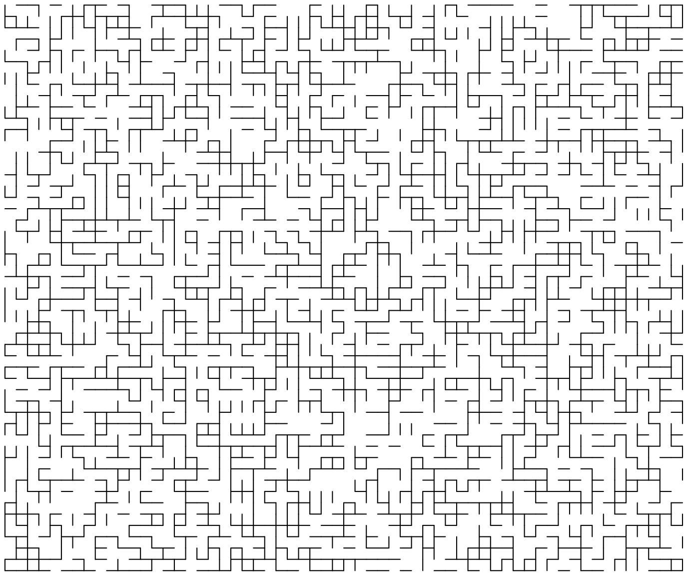

Figure 1.3: A realization of Bernoulli bond percolation with parameter $p = 0.5$ on a 60 by 50 window of the square lattice.

this parameter value will be critical, that is, will be on the boundary on two qualitatively different regimes. If $p > p_c$ then $\Theta(p) > 0$ and the standard 0-1 law implies that there is an infinite open component somewhere in the graph. It also turns out that this infinite cluster is unique in the sense that there is exactly one such cluster with probability 1. This result was first proved in [1] and [2]. Later the proof was simplified in [16] and extended in [25]. We call this regime the supercritical state. However, if $p < p_c$, called the subcritical state, similar argument implies that every open cluster in the graph is finite. The infinite open cluster naturally looks somewhat different for different supercritical parameters, namely it becomes thicker as p increases. Still, a number of its characteristics are similar for all values of p in the supercritical regime. Similar behaviour can be observed in the subcritical state.

The first natural question concerning the critical probability is whether it is trivial, i.e. it is equal to 0 or 1. If the simplest possible lattice, $\mathbb{Z}$, is considered then this indeed is the case with $p_c = 1$. The study of percolation models when the critical probability is trivial is uninteresting. With the exception of $\mathbb{Z}$, this does not happen on either of the above mentioned lattices, where we always have $0 < p_c < 1$. Usually the proof of this property is not that difficult. However, a much more involved question is the exact value of p_c. The most famous model where the answer is known is bond percolation on the square lattice, where $p_c = 1/2$; see [31]. Other such models are site percolation on any self-matching planar lattice (graphs that satisfy certain symmetry conditions; for instance the triangular lattice, see [47]) , bond percolation on the triangular and the honeycomb lattices; see [51]. Among the dependent percolation models, for instance p_c is known to be 1/2 in the divide and colour model on the triangular lattice, see [7], and for continuum percolation an example is the Voroni model with again $p_c = 1/2$; see [11].

This thesis mostly concerns itself with critical or near critical models. Therefore, our main interest in percolation lies in the $p = p_c$ case. For a long time only very little was known about critical percolation although it was widely conjectured to have the properties of critical models mentioned in the previous section. The first question concerning the critical model could be whether there is percolation, i.e. there is an infinite open cluster, at criticality. This question can be reformulated in terms of the percolation function. We know that $\Theta(p) = 0$ if $p < p_c$ and $\Theta(p) > 0$ and continuous if $p > p_c$. This latest fact is also non-trivial; see Section 8.3 in [26]. Therefore, a positive answer of the previous question follows if $\Theta(p)$ is continuous for every $p \in [0, 1]$. The answer is known for $\mathbb{Z}^d$ only for $d = 2$, see [29], and $d \geq 19$, see [28], and it is affirmative in all these cases.

As we mentioned earlier, the two main characteristics of critical systems are the power laws and the scale-free behaviour. In the case of percolation, physicists have predicted power laws for the distribution of the size of clusters and other quantities but a rigorous mathematical proof was lacking. The breakthrough was made about ten years ago by the introduction of the stochastic Loewner evolution, and the proof of conformal invariance that made it possible to compute different power law exponents. See [50] for a comprehensive treatment of these results. Furthermore, interfaces between open and closed bonds on all scales can be studied and shown in [17] to have fractal like structure.

Finally, we would like to give an alternative definition of the Bernoulli percolation that will be later used in this thesis. To each edge in $\mathbb{Z}^2$ we attach a random variable $\mathcal{U}_e$ that is uniformly distributed on $[0, 1]$. These variables are independent for different edges. For any $p \in [0, 1]$ we call an edge e p-open if and only if $\mathcal{U}_e < p$. It is immediate that the probability of an edge being p-open is p and the random graph induced by the p-open edges has the same distribution as a configuration of open edges in a Bernoulli percolation with parameter p. The advantage of this definition is that it couples all Bernoulli percolation models. We will make use of this definition in Chapter 3 when studying the invasion percolation process.

The models considered in this thesis are all connected to independent percolation, which explains that many techniques and theorems from Bernoulli percolation are used in our proofs. A list of theorems necessary to understand the coming chapters is presented in the appendix.

1.1.3 Self-destructive percolation

The first model considered in this thesis arises from studying forest-fire models introduced by Drossel and Schwabl in [22]. Let $B(n)$ be the subgraph of $\mathbb{Z}^2$ that is within the box $[-n, n]^2$. $B(n)$ will be the "land where our forest grows". Trees can grow at the sites of $B(n)$. Initially the land is bare, i.e. all sites of $B(n)$ are vacant. Every site has a Poisson clock attached to it with parameter 1, independent of the clocks of the other sites. When the clock rings, a tree wants to grow at the site. If a tree is already present nothing happens, if not the site becomes occupied. As the name of the model suggests, occasionally fire takes place in the forest. The fires are also governed by Poisson clocks but the burning mechanism is more complicated than that of the growth. Each site v has another Poisson clock with parameter λ, independent of the clocks of the other sites and the "growth-clock" of v. When this clock rings, a lightning strikes at the site. If there is no tree present, nothing happens. However, when there is a tree at the site it is set on fire that spreads to the whole cluster of the tree. Trees are burnt down instantaneously so when a tree of a cluster is hit, the whole cluster gets destroyed immediately.

Forests typically consist of a very large number of trees, therefore in our model it is reasonable to take n large. As mathematicians tend to do, we take the limit as n goes to infinity. If a forest is large a few trees burning down makes little difference. Hence, we are interested only in fires that consume large cluster. Furthermore, it is logical that lightnings happen much more often than growth of a tree. In the mathematical language these translate to taking the limit as $\lambda \downarrow 0$. Note that it is important to take the limits in the proper order. If we take first $\lambda \downarrow 0$ in a finite box, then with probability 1 there will be no lightning ever and trees will just grow uninterrupted till all sites are occupied. Therefore, if we take the limit in λ first and then as $n \to \infty$ we end up with an uninteresting model (that involves only growth of trees and no fires). On the other hand, if we take $n \to \infty$ first and the $\lambda \downarrow 0$ we may intuitively argue that the limiting process will have the following properties. Since $\lambda \downarrow 0$, the probability of any finite cluster being hit in any finite length of time goes to 0. Therefore, we can expect that with probability 1 no finite cluster is hit in the limit. However, if for any $\lambda > 0$ an infinite cluster arises in the limit as $n \to \infty$, it is destroyed immediately. A property which we may expect to be preserved as $\lambda \downarrow 0$. We may therefore argue that the limiting process will be very similar to the permanent self-destructive process introduced in [9]. For the permanent self-destructive model consider $\mathbb{Z}^2$ with all sites being vacant initially. Each site becomes occupied at rate 1, independently. As soon as an infinite cluster appears, it is destroyed immediately, that is all the vertices in the cluster are changed to vacant again. Intuition suggests that this model exhibits self-organized criticality. The behaviour of the process is clear up to time t_c corresponding to $p_c = 1 - e^{-t_c}$, where p_c is the critical probability corresponding to Bernoulli percolation on $\mathbb{Z}^2$. In the interval $[0, t_c]$ there is no infinite cluster present, therefore nothing is destroyed. But directly after t_c such a cluster is formed and instantaneously destroyed. If self-organized critical behaviour indeed occurs, then right after this moment another infinite cluster is formed and subsequently destroyed. Therefore, after time t_c the system will be in a permanent critical state and in each moment a new cluster is formed and destroyed. We would like to emphasize that there is *no proof* of SOC in the permanent self-destructive model and this intuitive argument serves only to show that we *may* expect SOC to occur. In a recent paper Ráth and Tóth [45] prove SOC in a closely related but simplified, so-called mean-field model on Erdős-Rényi random graphs.

In order to gain insight to the permanent self-destructive process a two-step model was introduced in [9] that models the destruction of one infinite cluster and the evolution of the configuration after the destruction has taken place. This model is called self-destructive percolation (SDP). The underlying lattice is still $\mathbb{Z}^2$ and first a Bernoulli site percolation with parameter p is performed where $p > p_c$, hence the configuration contains an infinite cluster. Next, this cluster is removed, that is all of its vertices are changed to vacant. Finally, each site is given a δ enhancement: each site that is closed becomes open with probability δ, independently. The two parameters of the model are p and δ. Note that only one destruction takes place in the SDP model. Furthermore, the process can also be defined for $p \leq p_c$. But then its behaviour is simple. Since there is no infinite cluster in the p-percolation, nothing is removed and the distribution of the final configuration is just a product measure with the appropriate parameter depending on p and δ.

As in the case of independent percolation, we can again ask if an infinite cluster exists in the final configuration. The probability of that event is denoted by $\Theta(p, \delta)$, in line with the notation for independent percolation. In [9], a conjecture is formulated for the self-destructive model with remarkable consequences for the existence of the permanent self-destructive percolation process. This conjecture involves the continuity of $\Theta(p, \delta)$ at the point $(p_c, 0)$. Unfortunately, we have not been able to prove or disprove the conjecture but it naturally gives rise to study the continuity of $\Theta(p, \delta)$, which requires developing new techniques to tackle the dependencies introduced by the self-destruction mechanism. We discuss and prove several properties of the SDP model in Chapter 2.

1.1.4 Invasion percolation

The second model considered in this thesis is called invasion percolation. An intuitive introduction to this model is as follows. Consider an infinite piece of land that is divided into square parcels. On all four sides of each parcel there is a dike. The heights of the dikes are distributed uniformly on $[0, 1]$ independently of each other. One of the parcels contains an infinite source of water. First, the water level in the parcel of the origin rises until it reaches the height of the lowest adjacent dike and then it spills over into the parcel on the other side of this dike. Next, the water level rises in both parcels until it reaches the height of the lowest dike on the boundary of the union of the two parcels, at which time a new parcel floods. The process continues indefinitely, and as time approaches infinity, an infinite region of land will flood. We are interested in the properties of the flooded region at time infinity.

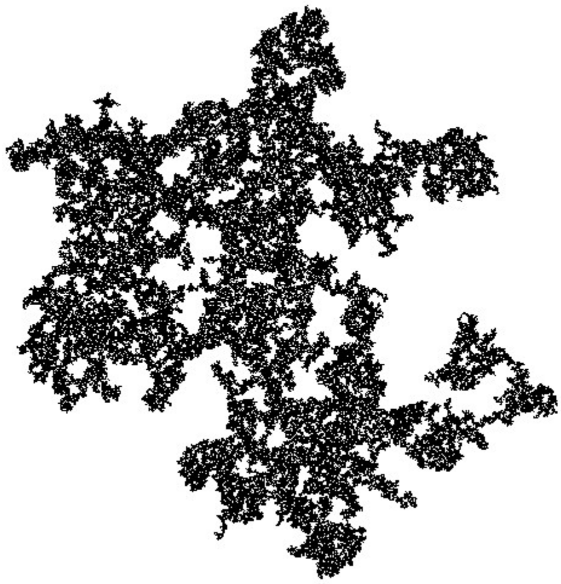

Figure 1.4: Invaded region at time 1 million. We thank Prof. Jon Machta from the University of Massachusetts for his permission to use this picture.

The following description is more mathematical. The complete formal definition of the model will be presented in Chapter 3. Consider $\mathbb{Z}^2$ as underlying lattice and assign random variables τ_e to each edge e, distributed uniformly on $[0, 1]$, independently of each other. We inductively define an increasing sequence $G_0, G_1, \ldots$ of connected subgraphs of the lattice. The first graph, G_0, consists of the origin only. At the first step we find the edge with the lowest τ emanating from the origin and add it and the corresponding vertex to G_0 to obtain G_1. We basically repeat this procedure. At step i we find the edge on the boundary of G_{i-1} that has the lowest τ value and add it to G_{i-1} along with the corresponding vertex to obtain G_i. The graph G_i is called the invaded region at time i, and the graph $\mathcal{S} = \cup_{i=0}^{\infty} G_i$ is called the *invasion percolation cluster* (IPC). The invaded region at time 1 million is pictured on Figure 1.4. The fractal-like structure of the invaded region is clearly visible.

At first sight it might be unclear why and in what sense this process shows self-organized critical behaviour. As a first indicator we mention that if e_n is the edge invaded at time n, then $\limsup_{n\to\infty} \tau_{e_n} = p_c$. Furthermore, one can define certain natural subgraphs of the IPC, called ponds, whose size show power law behaviour, at least given some widely believed conjectures about critical Bernoulli percolation clusters. Further similarities can be found when comparing the IPC to an object called the incipient infinite cluster. We will discuss this object in the next subsection but loosely speaking it is the “infinite cluster at criticality”. In Chapter 3 we discuss and prove such similarities and some surprising differences as well.

1.1.5 Dynamic incipient infinite cluster model

The last model considered in this thesis is not a real self-organized critical model because it requires fine- tuning of a parameter. Nevertheless, with a slight abuse of terminology we may call it organized criticality. It is a dynamic model that after tuning its parameter drives itself into a critical state. Naturally, constructing such a model is easy. For instance consider an arbitrary configuration of open and closed edges on $\mathbb{Z}^2$. Each edge has also a rate 1 Poisson clock attached to it, independently of each other. When the clock rings, the state of the bond is updated: it will be open with probability p_c and closed otherwise. If we let this dynamic run, eventually every edge in any finite box will have been updated at least once and the random graph of open edges will have the same distribution as critical Bernoulli percolation. The dynamic we consider in this thesis is more complicated, although is has similar flavour. We would like to create a dynamic model that approaches a more special distribution, namely that of the incipient infinite cluster.

As we mentioned earlier there is no percolation in $\mathbb{Z}^2$ at criticality, or by other words $\Theta(p_c) = 0$. Therefore we cannot talk about the infinite cluster at criticality. However one can consider the measure $\nu_n(.) = \mathbb{P}(\ .\ |\ 0 \leftrightarrow \partial B(n))$. For every n, the origin is connected to the boundary of $B(n)$ with ν_n-probability 1. Kesten showed in [33], that the limit

$$\nu(E) = \lim_{n\to\infty} \mathbb{P}(E|\ 0 \leftrightarrow \partial B(n))$$

exists for all event E that depends on the state of only finitely many edges. By standard arguments, ν has a unique extension to a probability measure on the configuration of open and closed edges and under this measure, the origin is in an infinite cluster with probability 1. This cluster is called the incipient infinite cluster (IIC). We would like to approximate the distribution of the IIC as the stationary measure of a dynamic process.

The formal definition of our process is rather technical and will be presented in Chapter 4. However, the underlying idea is easy to understand. Consider an arbitrary configuration of open and closed edges in $\mathbb{Z}^2$ where the origin is in an infinite open cluster. To each bond we attach a Poisson clock with some rate λ_e depending on the edge. The clocks are independent of each other. When a clock rings we try to update the corresponding edge. We say try because we would like to keep the origin in an infinite cluster at all times and, therefore, we cannot always change an open edge to closed. First, we have to check whether closing the edge disconnects the origin from infinity. If so we do not update. In this case, we say that the edge e is pivotal for the origin being connected to infinity, or simply that e is pivotal. If e is not pivotal, we update the state of the edge: it will become open with probability p_c and closed otherwise. We call this process the dynamic IIC process. It is important to emphasize that in spite of the simple definition the process does not necessarily exist. We will here provide an informal argument why the existence of the process is not obvious. In order to explain the problems that may arise, let all rates be 1. In this case, infinitely many edges want to update in any finite time interval. Now consider an edge e that wants to update at time t. Since we have Poisson clocks, with probability 1 no other edge want to update exactly at time t. Therefore, one may intuitively argue that the event that e is pivotal is the same as the event that there exist a $t_0 < t$ such that e is pivotal in the interval $(t_0, t]$. Note however, that since in any interval infinitely many edges want to update we may not be able to decide whether such t_0 exists or not and therefore we do not know if it is allowed to update e at time t. Clearly the existence of the process may very well depend on the rates. If for instance they decay exponentially fast in the distance between e and the origin, then with probability 1 only finitely many of them want to update in any finite interval and the above described problem cannot arise. In this case, it is easy to show that the process exists.

In Chapter 4, we propose to study the above described dynamic model. We give several possible ways to define the process with mathematical rigour and we formulate some open questions that are topic of our future work.

1.1.6 List of publications

1. J. van den Berg, R. Brouwer and B. Vágvölgyi. Box-crossings and continuity results for self-destructive percolation in the plane. *In and Out of Equilibrium 2. Progress in Probability, Vol 60*, pp 117–135, Birkháuser Verlag, Basel, Switzerland, 2008
2. J. van den Berg, A.A. Járai and B. Vágvölgyi. The size of a pond in $2D$ invasion percolation. *Electr. Comm. Prob. 12:* 411–420, 2007
3. M. Damron, A. Sapozhnikov and B. Vágvölgyi. Relations between invasion percolation and critical percolation in two dimensions. *To appear in The Annals of Probability*, 2009
4. A.A. Járai and B. Vágvölgyi. Stochastic dynamics for the incipient infinite cluster. *Work in progress.* 2009

Chapter 2

Self-Destructive Percolation

2.1 Introduction and outline of results

2.1.1 Background and motivation

In this chapter we will discuss our results on self-destructive percolation (abbreviated as SDP model) on the square lattice. Thus, unless explicitly indicated otherwise, the underlying lattice is always $\mathbb{Z}^2$. The model is described as follows: First we perform independent site percolation on this lattice: we declare each site *occupied* with probability p, and *vacant* with probability $1-p$, independent of the other sites. We will use the notation $\{V \leftrightarrow W\}$ for the event that there is an occupied path from the set of sites V to the set of sites W. We write $\{V \leftrightarrow \infty\}$ for the event that there is an infinite occupied path starting at V.
Let, as usual, $\theta(p)$ denote the probability that a given site, say $O = (0,0)$, belongs to an infinite occupied cluster. It is known that there is a critical value $0 < p_c < 1$ such that $\theta(p) > 0$ for all $p > p_c$, and $\theta(p) = 0$ for all $p \leq p_c$. Now suppose that, by some catastrophe, the infinite occupied cluster (if present) is destroyed; that is, each site in this cluster becomes vacant. Further suppose that after this catastrophe we give the sites independent 'enhancements', as follows: Each site that was already vacant in the beginning, or was *made* vacant by the catastrophe, becomes occupied with probability δ, independent of the others. Let $\mathcal{P}_{p,\delta}$ be the distribution of the final configuration.

A more formal, and often very convenient description of the model is as follows: Let X_i, $i \in \mathbb{Z}^2$ be independent $0-1$ valued random variables, each X_i being 1 with probability p and 0 with probability $1-p$. Further, let Y_i, $i \in \mathbb{Z}^2$, be independent $0-1$ valued random variables, each Y_i being 1 with probability δ and 0 with probability $1-\delta$. Moreover, we take the collection of Y_i's independent of that of the X_i's. Let $X_i^*, i \in \mathbb{Z}^2$ be defined by

$$X_i^* = \begin{cases} 1 \text{ if } X_i = 1 \text{ and there is no } X\text{- occupied path from } i \text{ to } \infty \\ 0 \text{ otherwise,} \end{cases} \tag{2.1.1}$$

where by 'X-occupied path' we mean a path on which each site j has $X_j = 1$. Finally, define $Z_i = X_i^* \vee Y_i$. This collection $(Z_i, i \in \mathbb{Z}^2)$ is (with 0 meaning 'vacant' and 1 'occupied') what we called 'the final configuration', and the above mentioned $\mathcal{P}_{p,\delta}$ is its distribution.

We use the notation $\theta(p, \delta)$ for the probability that, in the final configuration, O is in an infinite occupied cluster:

$$\theta(p, \delta) := \mathcal{P}_{p,\delta}(O \leftrightarrow \infty).$$

Note that O is occupied in the final configuration if and only if the above mentioned enhancement was successful, or O belonged initially (before the catastrophe) to a non-empty but finite occupied cluster. This gives

$$\mathcal{P}_{p,\delta}(O \text{ is occupied }) = \delta + (1 - \delta)(p - \theta(p)).$$

Also note that, in the case that $p \leq p_c$, nothing happens in the above catastrophe, so that in the final configuration the sites are independently occupied with probability $p + (1 - p)\delta$. Formally, if $p \leq p_c$, then

$$\mathcal{P}_{p,\delta} = \mathcal{P}_{p+(1-p)\delta}, \tag{2.1.2}$$

where we use the notation $\mathcal{P}_p$ for the product measure with parameter p. In particular,

$$\theta(p_c, \delta) = \theta(p_c + (1 - p_c)\delta) > 0, \tag{2.1.3}$$

for each $\delta > 0$.

Remark 2.1.1. *Most of what we said above has straightforward analogs for arbitrary countable graphs, but there are subtle differences. For instance, on the cubic lattice it has not yet been proved that $\theta(p_c) = 0$ (although this is generally believed to be true). So, for that lattice,* (2.1.2) *with $p = p_c$, and hence* (2.1.3)*, are not rigorously known.*

It is also clear from the construction that $\mathcal{P}_{p,\delta}$ stochastically dominates $\mathcal{P}_\delta$. Hence, if $\delta > p_c$ then $\theta(p, \delta) \geq \theta(\delta) > 0$ for all p.

It turns out that, if $p > p_c$, a 'non-negligible' enhancement is needed after the catastrophe to create again an infinite occupied cluster.

Proposition 2.1.2 ([9], Proposition 3.1)**.** *Suppose that $p > p_c$. Then $\exists \delta$ such that there is no infinite cluster in the final configuration of the SDP model with parameters p and δ. In particular, if $p(1 - \delta) \geq p_c$, then $\Theta(p, \delta) = 0$*

Proof. Following [9], suppose we colour each site i that have $X_i = 1$ and $Y_i = 0$ red. The probability of a site being red is therefore $p(1 - \delta)$ independently of each other. We assumed that $p(1 - \delta) \geq p_c$ and hence the RSW theorem implies that there exist infinitely many red circuits surrounding the origin. Since $p > p_c$ these red circuit will be part of the infinite $X-$cluster from some point on and therefore they will be destroyed by the catastrophe. Furthermore, since $Y_i = 0$ on these sites, the δ enhancement will not make these sites occupied again. So they will be closed in the final configuration and since they surround the origin, it cannot be in an infinite cluster, i.e. $\Theta(p, \delta) = 0$ $\square$

Remark 2.1.3. *The previous proposition also holds for the SDP model on the hypercubic lattice $\mathbb{Z}^d$ but the proof is much more involved. For details see Proposition 2.3.4. in [15].*

A much more difficult question is whether the needed enhancement goes to 0 as $p \downarrow p_c$. By (2.1.3) one might be tempted to reason intuitively that this is indeed the case. In [9] it was shown that for the analogous model on the binary tree this is correct. However, in [9] a different behaviour is conjectured for the square lattice.

Conjecture 2.1.4. *For self-destructive percolation on the square lattice there exists a $\delta > 0$ such that*

$$\forall p > p_c, \qquad \Theta(p, \delta) = 0. \tag{2.1.4}$$

In particular, there exists a $\delta > 0$ such that the function $\Theta(p, \delta)$ is not continuous on the line-segment $\{p_c\} \times (0, \delta)$.

In [9], this conjecture is presented as a consequence of a more general conjecture. Let us define a finite version of the SDP model. Let $G(n) = [-n, 3n] \times [-n, 2n]$ and $R(n) = [0, 2n] \times [0, n]$. First, we perform independent site percolation on $G(n)$ with parameter p_c. In the destruction step we remove each vertex that has an open path to the boundary of $G(n)$. Finally, we give a δ enhancement to each site in $G(n)$. Let us denote the distribution of the resulting random graph by $\mathbb{P}^{[G(n)]}_{p_c,\delta}$ and define

$$p_n(\delta) = \mathbb{P}^{[G(n)]}_{p_c,\delta}(t(n) \leftrightarrow b(n) \text{ inside } R(n)), \tag{2.1.5}$$

where $t(n) = [0, 2n] \times \{n\}$ is the top and $u(n) = [0, 2n] \times \{0\}$ is the bottom side of $R(n)$. See Figure 2.1 for a picture of the above defined event.

It is immediate from the definition that $p_n(\delta)$ is increasing in δ so we can define

$$\hat{\delta_c} = \sup\{\delta : p_n(\delta) \text{ is bounded away from 1, uniformly in } n\}. \tag{2.1.6}$$

Conjecture 2.1.5. *([9], Conjecture 3.2) $\hat{\delta_c} > 0$.*

Conjecture 2.1.5 turns out to have remarkable consequences both for the SDP model and for the permanent self-destructive percolation process.

Theorem 2.1.6. *([9], Theorem 3.3). If Conjecture 2.1.5 holds then so does Conjecture 2.1.4.*

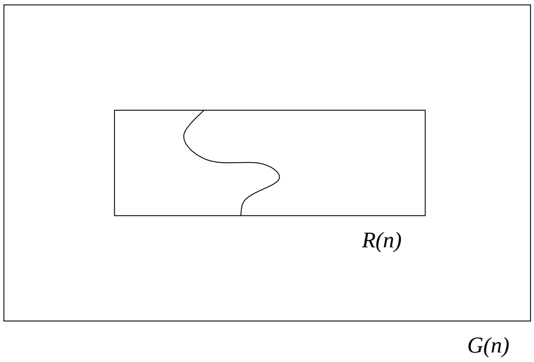

Figure 2.1: The event used to define $p_n(\delta)$ and consequently $\hat{\hat{\delta}}_c$

In order to state that result for the permanent self destructive percolation we first need to give a proper mathematical definition of the model. Following [9], we define:

Definition 2.1.7. *A collection of stochastic processes $\chi_i : \mathbb{R}_+ \to \{0,1\},\ i \in \mathbb{Z}^2$ is called the permanent self-destructive percolation process if it satisfies the following four properties:*

(i) Almost surely, for all $i \in \mathbb{Z}^2,\ \chi_i(0) = 0$.

(ii) Almost surely, for all $i \in \mathbb{Z}^2, t \mapsto \chi_i(t)$ is continuous from the right having left limits (cadlag).

(iii) Let $U_i^{(k)}$ be the length of the kth time interval during which $\chi_i(.)$ equals to 0. Then the random variables $\{U_i^{(k)}\}_{i\in\mathbb{Z}^2,k\in\mathbb{N}}$ are independent, exponentially distributed with parameter 1.

(iv) Almost surely for all $t \in \mathbb{R}_+$ and $i \in \mathbb{Z}^2$ with $\chi_i(t^-) = 1$; if there is an infinite path π from i having $\chi_i(t^-) = 1$ for all j on π then $\chi_i(t) = 0$, else $\chi_i(t) = 1$.

Theorem 2.1.8. *If Conjecture 2.1.5 holds then there are no processes $\chi_i : \mathbb{R}_+ \to \{0,1\},\ i \in \mathbb{Z}^2$ satisfying the four defining properties of the permanent self-destructive percolation.*

Unfortunately we have not been able to prove or disprove Conjecture 2.1.5 and it has only been supported by numerical simulations. Furthermore, it is another interesting and open problem whether Conjecture 2.1.4 implies Theorem 2.1.8.

Nonetheless, Conjecture 2.1.4 naturally raises the question whether the function $\Theta(.,.)$ is continuous in the complement of a region of the above mentioned form: is there a $\delta > 0$ such that $\theta(\cdot,\cdot)$ is continuous outside the set $\{p_c\} \times [0,\delta]$? In the next subsection we state that this is indeed the case, and give a summary of the methods and intermediate results used in the proof. At the end of Section 2.6 we point out why our proof does not work at points (p_c, δ) with small δ. We hope our arguments provide a better understanding of the earlier mentioned conjecture and will trigger new attempts to prove (or disprove) it.

2.1.2 Outline of results

The conjecture mentioned in the previous subsection raises the natural question whether $\theta(\cdot,\cdot)$ *is* continuous outside the indicated 'suspected' region. The following theorem states that this is indeed the case.

Theorem 2.1.9. *There is a $\delta \in (0,1)$ such that the function $\theta(\cdot,\cdot)$ is continuous outside the segment $\{p_c\} \times (0,\delta)$.*

As could be expected, the proof widely uses tools and results from ordinary percolation. However, the dependencies introduced by the self-destructive mechanism cause complications. Until recently, a serious obstacle was the absence of a suitable RSW-like theorem. This obstacle could be removed by the use of (a modified and somewhat stronger form of) a recent theorem of Bollobás and Riordan ([11]).

A rough outline of the proof of Theorem 2.1.9, and the needed intermediate results that are interesting in themselves, is as follows: In section 2 we list some basic properties of our model, which will be used later. The results in Section 3, which are also contained in the recent PhD thesis [15] of one of us, show that if $\theta(\cdot,\cdot)$ is strictly positive in some open region, then it is continuous on this region. It is also shown that if $\theta(p,\delta) = 0$, then $\theta(\cdot,\cdot)$ is continuous at (p,δ). These two results reduce the proof of Theorem 2.1.9 to showing that if $\theta(p,\delta) > 0$ and $p \neq p_c$, then $\theta(p,\delta) > 0$ in an open neighborhood of (p,δ). This in turn requires a suitable finite-size criterion (see below) for SDP. In Section 4 we give the modified form of the Bollobás-Riordan theorem. This is used in Section 5 to obtain the above mentioned finite-size criterion. Finally, in Section 6 we combine these results and prove the main theorem.

We end this section with the following remark: When we say that a function f is 'increasing' ('decreasing') this should, unless this is preceded by the word 'strictly', be interpreted in the weak sense: $x < y$ implies $f(x) \leq f(y)$.

2.2 Basic properties

In this section we state some basic properties which will be used later.

First some more terminology and notation: If $v = (v_1, v_2)$ and $w = (w_1, w_2)$ are two vertices, we let $|v - w|$ denote their (graph) distance $|v_1 - w_1| + |v_2 - w_2|$. By $B(v,k)$ and $\partial B(v,k)$ we denote the set of vertices w for which $|v - w|$ is at most k, respectively equal to k. For $V, W \subset \mathbb{Z}^2$, we define the distance between V and W as $\min\{|v - w| : v \in V, w \in W\}$.

Recall that $\mathcal{P}_{p,\delta}$ denotes the SDP distribution (that is, the distribution of the collection $(Z_i,\, i \in \mathbb{Z}^2)$ defined in Subsection 1.1). This is a distribution on $\Omega := \{0,1\}^{\mathbb{Z}^2}$ (with the usual σ-field). Elements of Ω are typically denoted by $\omega (= (\omega_i, i \in \mathbb{Z}^2))$, σ etc. We write $\omega \leq \sigma$ if $\omega_i \leq \sigma_i$ for all i.
Let V be a set of vertices and A an event. We say that A lives on V if $\omega \in A$ and $\sigma_i = \omega_i$ for all $i \in V$, implies $\sigma \in A$. And we say that A is a cylinder event if A lives on some finite set of vertices. As usual, we say that A is increasing if $\omega \in A$ and $\omega_i \leq \sigma_i$ for all i, implies $\sigma \in A$. The first two lemma's below come from Section 2.2 and 2.4 respectively in [9].

Lemma 2.2.1. *Let A and B be two increasing cylinder events. We have*

$$\mathcal{P}_{p,\delta}(A \cap B) \geq \mathcal{P}_{p,\delta}(A)\mathcal{P}_{p,\delta}(B).$$

As to monotonicity, it is obvious that the SDP model has monotonicity in δ: If $\delta_1 \geq \delta_2$, then $\mathcal{P}_{p,\delta_1}$ stochastically dominates $\mathcal{P}_{p,\delta_2}$. Although there seems to be no 'nice' monotonicity in p we have the following property.

Lemma 2.2.2. *If $p_2 \geq p_1$ and $p_2 + (1-p_2)\delta_2 \leq p_1 + (1-p_1)\delta_1$, then*

$$\mathcal{P}_{p_1,\delta_1} \textit{ dominates } \mathcal{P}_{p_2,\delta_2}.$$

The next result is about 'almost independence' of cylinder events which live on widely separated sets. As usual, the lattice which has the same vertices as the square lattice but where each vertex has, besides the four edges to its nearest neighbours, also four 'diagonal edges' is called the matching lattice (of the square lattice). To distinguish paths and circuits in the matching lattice from those in the square lattice, we use the terminology *-paths and *-circuits.

Lemma 2.2.3. *Let k be a positive integer and let V and W be subsets of $\mathbb{Z}^2$ that have distance larger than $2k$. Further, let A and B be events which live on V and W respectively. Then*

$$\begin{aligned} &|\mathcal{P}_{p,\delta}(A \cap B) - \mathcal{P}_{p,\delta}(A)\mathcal{P}_{p,\delta}(B)| \leq 2(|V| + |W|) \\ &\mathcal{P}_p\left(\exists \textit{ vacant *-circuit surrounding } O \textit{ and some vertex in } \partial B(O,k)\right). \end{aligned} \tag{2.2.1}$$

Proof. Recall how we formally defined the SDP model in terms of random variables X, Y and Z. We use a modification of those variables: Let X and Y be as before, but in addition to X^* and Z we now define $X^{*(f)}$ and $Z^{(f)}$

by

$$
\begin{aligned}
X_i^{*(f)} &= \begin{cases} 1 \text{ if } X_i = 1 \text{ and } \nexists\ X\text{- occupied path from } i \text{ to } \partial B(i,k) \\ 0 \text{ otherwise;} \end{cases} \\
Z_i^{(f)} &= X_i^{*(f)} \vee Y_i. \qquad (2.2.2)
\end{aligned}
$$

Let $\mathcal{P}_{p,\delta}^{(f)}$ denote the distribution of $Z^{(f)}$. It is clear that the random variables $Z_i^{(f)}, i \in V$ are independent of the random variables $Z_i^{(f)}, i \in W$, and hence

$$\mathcal{P}_{p,\delta}^{(f)}(A \cap B) = \mathcal{P}_{p,\delta}^{(f)}(A)\mathcal{P}_{p,\delta}^{(f)}(B). \qquad (2.2.3)$$

Also note that if $Z_i \neq Z_i^{(f)}$, then the $X-$occupied cluster of i intersects $\partial B(i,k)$ but is finite. Hence there is an $X-$vacant circuit in the matching lattice that surrounds i and some site in $\partial B(i,k)$. Hence, since the $X-$ variables are Bernoulli random variables with parameter p, we have for any finite set K of vertices and any event E living on K,

$$
\begin{aligned}
&|\mathcal{P}_{p,\delta}(E) - \mathcal{P}_{p,\delta}^{(f)}(E)| \leq P(Z_K \neq Z_K^{(f)}) \leq \qquad (2.2.4) \\
&|K|\mathcal{P}_p\left(\exists \text{ a vacant *-circuit surrounding } O \text{ and some vertex in } \partial B(O,k)\right).
\end{aligned}
$$

The lemma now follows easily from (2.2.3) and (2.2.4) □

Our last result in this section is on the uniqueness of the infinite cluster.

Lemma 2.2.4. *If $\theta(p,\delta) > 0$, then*

$$\mathcal{P}_{p,\delta}(\exists \textit{ a unique infinite occupied cluster }) = 1.$$

Proof. We follow [15] with the proof. From the earlier construction of the SDP model in terms of the $X-$ and Y variables, it is clear that $\mathcal{P}_{p,\delta}$ is stationary and ergodic. It is also clear that in the SDP model the conditional probability that a given site is occupied given the configuration at all other sites, is at least δ. So this model has the so-called positive finite energy property. The result now follows from an extension in [25] of the well-known Burton-Keane ([16]) uniqueness result. □

2.3 Partial continuity results

In this section we first prove that in the SDP model the probabilities of cylinder events are continuous functions of (p,δ). Next we prove that the function

$\theta(\cdot,\cdot)$ is continuous at (p,δ) if $\theta(p,\delta)=0$ or there is an open neighbourhood of (p,δ) on which θ is strictly positive. Note that, once we have this, the proof of Theorem 2.1.9 is basically reduced to showing that if $p \neq p_c$ and $\theta(p_c,\delta)>0$, then $\theta(\cdot,\cdot)$ is strictly positive on an open neighbourhood of (p,δ).

Lemma 2.3.1. *Let A be a cylinder event. The function $(p,\delta) \to \mathcal{P}_{p,\delta}(A)$ is continuous on $[0,1]^2$.*

Remark 2.3.2. *The proof (see below) uses the well-known fact that $\theta(p_c)=0$. For many lattices (e.g. the cubic lattice) this fact has not been proved. For those lattices the arguments below show that the function in the statement of 2.3.1 is continuous on $[0,1]^2 \setminus (\{p_c\} \times [0,1])$. Related to this, Proposition 2.3.3 below would also need some modification for those lattices.*

Proof. Let A be an event which lives on some finite set V. Recall the construction of the SDP model in terms of random variables X, Y and Z. Let, for $\sigma \in \Omega$, σ_V denote the tuple $(\sigma_i, i \in V)$. It is clear that the distribution of X^*_V is a function of p only, and that, conditioned on X^*_V, the probability that $Z_V \in A$ is a polynomial (of degree $|V|$) in δ. Therefore it is sufficient to prove that, for each $\alpha \in \{0,1\}^V$, the function $f : p \to \mathcal{P}(X^*_V = \alpha)$ is continuous. Recall that the $X-$ variables are Bernoulli random variables (with parameter p). Now let $0 < p_1 < p_2$. In a standard way, by introducing independent, uniformly on the interval $(0,1)$ distributed random variables $U_i, i \in \mathbb{Z}^2$, we can suitably couple two collections of Bernoulli random variables with parameters p_1, respectively p_2. Such argument easily gives that $|f(p_2)-f(p_1)|$ is less than or equal to the sum over $i \in V$ of $\mathcal{P}(U_i \in (p_1,p_2)) + \mathcal{P}(i$ is in an infinite p_2-open but not in an infinite p_1-open cluster), which equals

$$|V|(p_2 - p_1 + \theta(p_2) - \theta(p_1)).$$

The lemma now follows from the continuity of $\theta(.)$. □

Proposition 2.3.3. *Let $(p,\delta) \in [0,1]^2$. If (a) or (b) below holds, the function $\theta(\cdot,\cdot)$ is continuous at (p,δ).*

(a) $\theta(\cdot,\cdot) > 0$ on an open neighbourhood of (p,δ).
(b) $\theta(p,\delta)=0$,

Proof. For this (and some other) results it is convenient to describe the SDP model in terms of Poisson processes: Assign to each site, independently of the other sites, a Poisson clock with rate 1. These clocks govern the following time evolution: Initially each site is vacant. Whenever the clock of a site rings, the site becomes occupied. (If it was already occupied, the ring is ignored). Note that if occupied sites would always remain occupied, then for each time t, the configuration at time t would be a collection of independent Bernoulli random variables with parameter $1-\exp(-t)$. In particular, before and at time t_c, defined by the relation $p_c = 1-\exp(-t_c)$, there would be no infinite occupied cluster, but after t_c there would be a (unique) infinite cluster. However, we do allow occupied sites to become vacant, although only once, as follows: Fix a time τ, a parameter of the time evolution. At time τ all sites in the infinite occupied cluster become vacant. (If there is no infinite occupied cluster, which is a.s. the case if $\tau \leq t_c$, nothing happens). After time τ we let the evolution behave as before; that is, each vacant site becomes occupied when its Poisson clock rings. Let, for this time evolution with parameter τ, $\hat{\mathcal{P}}_{\tau,t}$ denote the distribution of the configuration at time t , and let

$$\hat{\theta}(\tau, t) = \hat{\mathcal{P}}_{\tau,t}(O \text{ is in an infinite occupied cluster }). \tag{2.3.1}$$

It is easy to see that

$$\hat{\mathcal{P}}_{\tau,t} = \mathcal{P}_{p,\delta}, \tag{2.3.2}$$

where $p = 1 - \exp(-\tau)$ and $\delta = 1 - \exp(-(t - \tau))$. It is also easy to see that $\hat{\mathcal{P}}_{\tau,t}$ is stochastically decreasing in τ and stochastically increasing in t. In fact this is the key behind Lemma 2.2.2.

Now we come back to the proof of Proposition 2.3.3. From (2.3.1) and (2.3.2) we get (since the map between pairs (p, δ) and (τ, t) in (2.3.2) is continuous) that this proposition is equivalent to saying that if $\hat{\theta}(\tau, t) = 0$ or $\hat{\theta}$ is strictly positive on an open neighbourhood of (τ, t), then $\hat{\theta}$ is continuous at (τ, t). To prove this equivalent form of Proposition 2.3.3 we use ideas from [10]. The introduction of pairs (τ, t) as replacement of (p, δ) not only has the advantage that, as we already saw, we now have a more suitable form of monotonicity, but, more importantly, that we now have a more 'detailed' structure (the Poisson processes) in the background which gives the appropriate 'room' needed to get a suitable modification of the arguments in [10].

Let (τ, t) be as above. Divide the parameter space in four 'quadrants', numbered I to IV:

$$\begin{aligned}
I &:= [0, \tau] \times [t, \infty), \\
II &:= [\tau, \infty) \times [t, \infty), \\
III &:= [\tau, \infty) \times [0, t], \\
IV &:= [0, \tau] \times [0, t].
\end{aligned}$$

Note that it is sufficient to prove that for each monotone sequence $(\tau_i, t_i)_{i \geq 0}$ that lies in one of the above quadrants and converges to (τ, t), one has

$$\lim_{i \to \infty} \hat{\theta}(\tau_i, t_i) = \hat{\theta}(\tau, t).$$

We handle each of the quadrants separately.

Quadrant I) This is easy and corresponds to the (easy) proof of right continuity of ordinary percolation: Let (τ_i) be a monotone sequence which converges from below to τ and let (t_i) be a monotone sequence which converges from above to t. Let A_n denote the event that there is an occupied path from O to $\partial B(O,n)$. By monotonicity and Lemma 2.3.1 we have that

$$\text{For each } i, \hat{\mathcal{P}}_{\tau_i,t_i}(A_n) \downarrow \hat{\theta}(\tau_i,t_i) \text{ as } n \to \infty; \tag{2.3.3}$$

$$\hat{\mathcal{P}}_{\tau,t}(A_n) \downarrow \hat{\theta}(\tau,t) \text{ as } n \to \infty; \tag{2.3.4}$$

$$\text{For each } n, \hat{\mathcal{P}}_{\tau_i,t_i}(A_n) \downarrow \hat{\mathcal{P}}_{\tau,t}(A_n) \text{ as } i \to \infty, \tag{2.3.5}$$

From these three statements it is easy to see that $\hat{\theta}(\tau_i,t_i)$ tends to $\hat{\theta}(\tau,t)$ as $i \to \infty$.

Quadrant III) Let (τ_i) be a monotone sequence which converges from above to τ and (t_i) a monotone sequence which converges from below to t. By the earlier monotonicity arguments, the sequence $\hat{\theta}(\tau_i,t_i)$ is increasing in i, and has a limit smaller than or equal to $\hat{\theta}(\tau,t)$. So for the situation where $\hat{\theta}(\tau,t) = 0$, the proof is done. Now we handle the other situation: we assume $\hat{\theta}$ is positive in an open neighbourhood of (τ,t). For this situation considerable work has to be done. Note that in the dynamic description given earlier in this section, the underlying Poisson processes were the same for each choice of the model parameter τ. This allows us (and we already used this to derive some monotonicity properties) to couple the models with the different τ_i's and τ.

Let, for $s < u$, $C_{s,u}$ denote the occupied cluster of site O at time u in the process with parameter s (that is, under the time evolution where the infinite occupied cluster is destroyed at time s). Further, we use the notation $\omega(s,u)$ for the configuration at time u in that model. It is also convenient to consider $\omega(u)$, the configuration at time u in the model where *no* destruction takes place. (So, $\omega_v(u), v \in \mathbb{Z}^2$, are independent $0-1$ valued random variables, each being 1 with probability $1 - \exp(-u)$). Again we emphasize that all these models are defined in terms of the same Poisson processes. From monotonicity (note that $C_{\tau_i,t_i} \subset C_{\tau_{i+1},t_{i+1}}$ for all i) it is clear that

$$\lim_{i\to\infty} \hat{\theta}(\tau_i,t_i) = \mathcal{P}(\exists\, i\, |C_{\tau_i,t_i}| = \infty),$$

and

$$\hat{\theta}(\tau,t) - \lim_{i\to\infty} \hat{\theta}(\tau_i,t_i) = \mathcal{P}(|C_{\tau,t}| = \infty, \forall i\, |C_{\tau_i,t_i}| < \infty). \tag{2.3.6}$$

So we have to show that the r.h.s. of (2.3.6) is 0. Fix a j with the property that $\hat{\theta}(\tau_j,t_j) > 0$. Such j exists by the condition we assumed for (τ,t). To show that the r.h.s. of (2.3.6) is 0, it is sufficient (and necessary) to prove the following claim:

Claim
Apart from an event of probability 0*, the event* $\{|C_{\tau,t}| = \infty\}$ *is contained in the event that there is a* $k > j$ *for which* $|C_{\tau_k,t_k}| = \infty$.

So suppose $|C_{\tau,t}| = \infty$. By our choice of j we may assume that $\omega(\tau_j,t_j)$ has an infinite occupied cluster, and by Lemma 2.2.4 that this cluster is unique. We denote it by I_j. If $O \in I_j$ we are done. From monotonicity and the uniqueness of the infinite cluster (see Lemma 2.2.4), we have $I_j \subset C_{\tau,t}$. Hence there is a finite path π from O to some site in I_j such that $\omega(\tau,t) \equiv 1$ on π. Since, a.s. there are no vertices whose clock rings exactly at time t or τ, we may assume that for every site v on π, (a) or (b) below holds:

(a) The clock of v rings in the interval (τ, t).

(b) $\omega_v(\tau) = 1$ but the occupied cluster of v in $\omega(\tau)$ is finite.

If (a) occurs we define:

$$i_v := \min\{i \,:\, i \geq j \text{ and the clock of } v \text{ rings in } (\tau_i, t_i)\}.$$

Note that then, by the monotonicity of the sequence (τ_i, t_i), the clock of v rings in the interval (τ_l, t_l) for all $l \geq i_v$. If (a) does not occur, (b) occurs, and hence there is a finite set K_v of sites on which $\omega(\tau) = 0$ and which separates v from ∞. Then we use the following alternative definition of i_v:

$$i_v := \min\{i \,:\, i \geq j \text{ and } \omega(\tau_i) \equiv 0 \text{ on } K_v\}.$$

This minimum exists since K_v is finite and (again) we assume that no Poisson clock rings exactly at time τ. Now let

$$k := \max_{v \in \pi} i_v,$$

which exists since π is finite.

From the above procedure it is clear that $\omega_v(\tau_k, t_k) = 1$ for all v on π. Further, since $k \geq j$ and by monotonicity, also $\omega_v(\tau_k, t_k) = 1$ for all $v \in I_j$. Since π is a path from O to I_j this implies that I_j is contained in C_{τ_k,t_k} and hence that $|C_{\tau_k,t_k}| = \infty$. This proves the Claim above.

Quadrants II) and IV)

The required results for these quadrants follow very easily from monotonicity and the above results for quadrants I and III: Let (τ_i, t_i) be a sequence in quadrant II that converges to (τ, t). We have, by earlier stated monotonicity properties,

$$\hat{\theta}(\tau_i, t) \leq \hat{\theta}(\tau_i, t_i) \leq \hat{\theta}(\tau, t_i).$$

Since the sequence $(\tau_i, t))$ lies in quadrant III and the sequence (τ, t_i) lies in quadrant I, the upper and lower bound both converge to $\hat{\theta}(\tau, t)$. This completes the treatment of quadrant II. Quadrant IV is treated in the same way. This completes the proof of Proposition 2.3.3 □

2.4 An RSW-type result

For our main result we need to prove that if the crossing probability of an n by n square goes to 1 as $n \to \infty$, then also the crossing probability of a (say)

$3n \times n$ rectangle in the 'difficult direction' goes to 1 as $n \to \infty$. Such (and stronger) results were proved for ordinary percolation in the late nineteen seventies by Russo, and by Seymour and Welsh, and therefore became known as RSW theorems (See Appendix A). In fact, Russo, and Seymour and Welsh proved much stronger statements. For instance, they showed that there is a strictly increasing continuous function $f : [0,1] \to [0,1]$ such that for each n the probability to cross a $3n \times n$ rectangle is larger than or equal to

$$f(\text{ the probability of crossing an } n \times n \text{ rectangle }).$$

Their proofs used careful conditioning on the lowest horizontal crossing in a rectangle, after which the area above that crossing was treated, and a new, vertical crossing in that area was 'constructed'. Such arguments work for ordinary percolation because there the above mentioned area can be treated as 'fresh' territory. However, they usually break down in situations where we have dependencies, as in the SDP model.

Recently, Bollobás and Riordan ([11]) made significant progress on these matters. For the so-called Voronoi percolation model they proved an RSW type result. That result is one of the main ingredients in their proof that the critical probability for Voronoi percolation equals 1/2 (which had been conjectured but stayed open for a long time). Although they explicitly proved their RSW type result only for the Voronoi model, their proof works (as they remark in their paper) for a large class of models. The result we needed is a little stronger than that of [11]. We think that this improvement is useful for many other models as well. The rest of this section is organised as follows. First we give a short introduction to Voronoi percolation. Then we state the above mentioned RSW-like theorem of [11] and show the modified proof to obtain the stronger version. Finally we state the analog for the SDP model and explain why the proof for the Voronoi model works for this model as well.

2.4.1 The Voronoi percolation model

We start with a brief description of the Voronoi percolation model. The (random) Voronoi percolation model is as follows: Let Z denote the (random) set of points in a Poisson point process with density 1 in the plane. This set gives rise to a random Voronoi tessellation of the plane: Assign to each $z \in Z$ the set of all $x \in \mathbb{R}^2$ for which z is the nearest point in Z. The closure of this set is called the (Voronoi) cell of z. It is known that (with probability 1) each Voronoi cell is a convex polygon, and that two cells are either disjoint or share an entire edge. In the latter case the two cells are said to be neighbours or adjacent. This notion of adjacency gives, in a natural way, rise to the notion of paths, clusters etc.

Now consider the percolation model where each cell, independently of everything else, is coloured black with probability p and white with probability $1 - p$. Based on analogies with ordinary percolation (in particular with the self-matching property of the usual triangular lattice) it has been conjectured for a long time that the critical value for this percolation model is 1/2: for $p < 1/2$ there is (a.s.) no infinite black cluster, but for $p > 1/2$ there is an infinite black cluster (a.s.). As we said before, this was recently proved rigorously by Bollobás and Riordan ([11]), and a key ingredient in their proof is an ingenious RSW-like result.

2.4.2 The RSW-like result for Voronoi percolation

As in [11] we define, for the Voronoi percolation model with parameter p, $f_p(\rho, s)$ as the probability that there is a horizontal black crossing of the rectangle $[0, \rho s] \times [0, s]$. The following is Theorem 4.1 in [11]

Theorem 2.4.1. *(Bollobás and Riordan) Let $0 < p < 1$ be fixed.*

$$\begin{aligned} \textit{If} \qquad & \liminf_{s\to\infty} f_p(1, s) > 0, \\ \textit{then} \qquad & \limsup_{s\to\infty} f_p(\rho, s) > 0 \textit{ for all } \rho > 0. \end{aligned} \tag{2.4.1}$$

Studying the proof we realised that the condition can be weakened, so that the following theorem is obtained:

Theorem 2.4.2. *Let $0 < p < 1$ be fixed.*

$$\begin{aligned} \textit{If} \qquad & \limsup_{s\to\infty} f_p(\rho, s) > 0 \textit{ for some } \rho > 0, \\ \textit{then} \qquad & \limsup_{s\to\infty} f_p(\rho, s) > 0 \textit{ for all } \rho > 0. \end{aligned}$$

As we shall show, this somewhat stronger Theorem 2.4.2 can be proved in almost the same way as Theorem 2.4.1. But see Remark 2.4.4 about the global structure of the proof.

First note that Theorem 2.4.2 is (trivially) equivalent to the following:

Theorem 2.4.3. *Let $0 < p < 1$ be fixed.*

$$\textit{If } \limsup_{s\to\infty} f_p(\rho, s) = 0 \textit{ for some } \rho > 0, \tag{2.4.2}$$

then

$$\limsup_{s\to\infty} f_p(\rho, s) = 0 \textit{ for all } \rho > 0. \tag{2.4.3}$$

This is the form we will prove, following (with some small changes) the steps in [11].

Proof. (Theorem 2.4.2 and 2.4.3). Since p is fixed we will omit it from our notation. In particular we will write f instead of f_p.

First we rewrite the condition (2.4.2) in Theorem 2.4.3: If $\limsup_{s\to\infty} f(\rho, s) = 0$ for some $\rho \leq 1$ then, since $f(\rho, s)$ is decreasing in ρ, this lim sup is 0 for all $\rho > 1$. Moreover, the well-known pasting techniques from ordinary percolation show easily that if $\limsup_{s\to\infty} f(\rho, s) > 0$ for some $\rho > 1$, then this lim sup is positive for all $\rho' > \rho$, and hence (using again monotonicity of f in ρ) for all $\rho > 1$. Equivalently, if $\limsup_{s\to\infty} f(\rho, s) = 0$ for some $\rho > 1$, then this limit equals 0 for all $\rho > 1$. Hence, the condition in Theorem 2.4.3 is equivalent to

$$\limsup_{s\to\infty} f_p(\rho, s) = 0 \text{ for all } \rho > 1, \tag{2.4.4}$$

and this is also equivalent to condition (4.2) in Section 4 of [11]:

$$\limsup_{s\to\infty} f_p(\rho, s) = 0 \text{ for some } \rho > 1, \tag{2.4.5}$$

We will assume (2.4.4) (or its equivalent form (2.4.5)) and show how, following basically the proof of Theorem 2.4.1, the equation in (2.4.3) can be derived from it for all $\rho > 1/2$. Then we make clear that, for each k, very similar arguments work for $1/k$ instead of $1/2$, which completes the proof of Theorem 2.4.3.

Remark 2.4.4. *In [11] Bollobás and Riordan prove their theorem by contradiction: They assume (as we do here)* (2.4.4) *above, and, moreover they assume condition* (2.4.1) *(see equation (4.1) in their paper). Then, after a number of steps (claims), they reach a contradiction, which completes the proof. However, most of these steps do not use the 'additional' assumption* (2.4.1) *at all. We found a 'direct' (that is, not by contradiction) proof, see below, more clarifying since it leads more easily to further improvements. Most of the steps (claims) in the proof are practically the same as the corresponding step in [11]. Nevertheless, to keep this thesis self-contained we give a full proof of all the claims.*

First some notation and terminology: T_s is defined as the strip $[0, s] \times \mathbb{R}$. An event is said to hold *with high probability*, abbreviated *whp* if its probability goes to 1 as $s \to \infty$ (and all other parameters, e.g. p and ϵ are fixed). We say that a path P is a horizontal crossing of a rectangle R if P lies entirely within R and connects some point on the left-hand side of R to another point on the right-hand side of R. Vertical crossings are defined analogously. Finally, we say that a rectangle is s by ρs with some $\rho > 0$ if it has width s and height ρs.

Claims 1–4 are exactly the same as Claim 4.3–4.6 in [11], except that in their formulation not only (2.4.4) but also (2.4.1) is assumed. However, their proof do not use the latter assumption.

Claim 1 ([11], Claim 4.3): *Let $\varepsilon > 0$ be fixed, and let L be the line-segment $\{0\} \times [-\varepsilon s, \varepsilon s]$. Assuming that (2.4.4) holds, the probability that there is a black path P in $T_s = [0, s] \times \mathbb{R}$ starting from L and going outside $S' = [0, s] \times [-(1/2 + 2\varepsilon)s, (1/2 + 2\varepsilon)s]$ tends to 0 as $s \to \infty$.*

Proof. By symmetry, it is enough to show that *whp* there is no black path lying entirely within T_s and connecting L to the line $y = (1/2+2\varepsilon)s$. Denote by E the event that such a path exists. Furthermore, let E_1 be the subset of E that this path stays within $[0, s] \times [-s/2, (1/2 + 2\varepsilon)s]$. Note that if E holds but E_1 does not, then there is a vertical black crossing in the rectangle $[0, s]\times[-s/2, (1/2+2\varepsilon)s]$. This event has probability $f(1+2\varepsilon, s)$, which tends to 0 as s goes to infinity by (2.4.4). Therefore, it is enough to consider the event E_1.

Let $\tilde{L} = \{0\} \times [\varepsilon s, 3\varepsilon s]$ and let E_2 be the reflection of E_1 on the line $y = \varepsilon s$, namely E_2 is the event that there is a black path lying entirely within $[0, s] \times [-s/2, (1/2 + 2\varepsilon)s]$ and connecting some point in $\tilde{L}$ to the line $y = -s/2$. By symmetry, $\mathbb{P}(E_1) = \mathbb{P}(E_2)$ and they are both increasing events. An analogy of Lemma 2.2.1 also holds for the Voronoi setting, see Lemma 3.3 in [11]. Therefore, we have $\mathbb{P}(E_1 \cap E_2) \geq \mathbb{P}(E_1)^2$. But if $E_1 \cap E_2$ holds then there is a vertical crossing of $[0, s] \times [-s/2, (1/2 + 2\varepsilon)s]$ as pictured on Figure 2.2. The probability of having such a crossing is $f(1 + 2\varepsilon, s)$, which tends to zero by (2.4.4), proving the claim. □

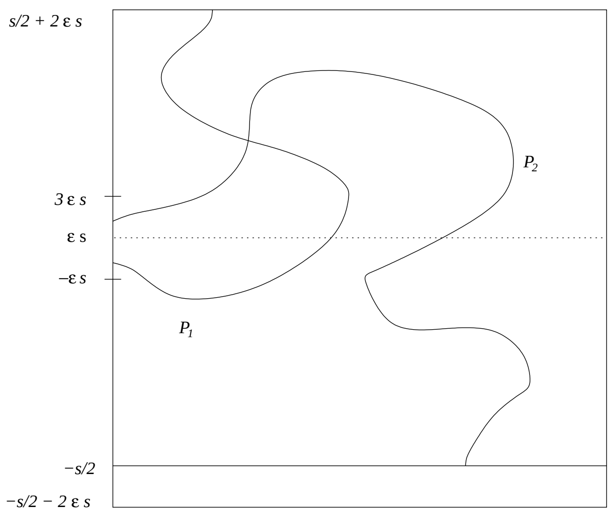

Figure 2.2: Events E_1 and E_2 as defined in the proof of Claim 1. Note that the paths P_1 and P_2, corresponding respectively to E_1 and E_2, must intersect.

The above, quite innocent looking claim leads step by step to stronger and eventually very strong claims. For a path $P \subset \mathbb{R}^2$, let $y_+(P)$ be the maximum "height" attained by the path, that is, the supremum of the y–

coordinates of the points on P. Similarly, let $y_-(P)$ be the infimum of the y–coordinates on P. Finally, if P is a crossing of a rectangle R we write $y_0(P)$ and $y_1(P)$ for the y–coordinates of the left and right endpoints of P, respectively.

Claim 2 ([11], Claim 4.4): *Let $\varepsilon > 0$ and $C > 0$ be fixed and R_s be an s by $2Cs$ rectangle. Assuming* (2.4.4)*,* whp *every black horizontal crossing P of R_s satisfies*

$$\big||y_\pm(P) - y_i(P)| - s/2\big| \leq \varepsilon s, \tag{2.4.6}$$

for $y_\pm(P) = y_+(P),\ y_-(P)$ and $i = 0, 1$. In particular

$$|y_0(P) - y_1(P)| \leq 2\varepsilon s. \tag{2.4.7}$$

Even though this claim may look plausible for the first sight, note that it requires a crossing to have very special structure. Consider a crossing P of the strip $[0, s] \times \mathbb{R}$. Without loss of generality we can assume that P starts from the point $(0, 0)$. Claim 2 then implies that this crossing ends within a 2ε neighbourhood of $(s, 0)$, the "highest" point occurs in the rectangle $[0, s] \times [s/2 - \varepsilon, s/2 + \varepsilon]$ and the "lowest" in $[0, s] \times [-s/2 - \varepsilon, -s/2 + \varepsilon]$; see Figure 2.3. Recall that we would like to prove that given (2.4.4) the probability of crossing of rectangles tends to zero and Claim 2 already points into that direction.

Proof. Applying (2.4.6) twice yields (2.4.7), therefore it is enough to prove (2.4.6). First note that due to symmetry, it is sufficient to show that *whp*

$$y_0(P) + s/2 - \varepsilon s \leq y_+(P) \leq y_0(P) + s/2 + \varepsilon s. \tag{2.4.8}$$

Without loss of generality we can take $R_s = [0, s] \times [-Cs, Cs]$. We omit P from our notation whenever it is clear from the context which path is considered. Furthermore, let us define E as the event that every black path starting from $\{0\} \times [-Cs, Cs]$ that crosses $T_s = [0, s] \times \mathbb{R}$ satisfies (2.4.8). The event E is contained in the event that is in our interest, so it is sufficient to show that E holds *whp*. We can cover the left-hand side of R_s by $O(1)$ line-segments L_i of length $\varepsilon s/2$. Let E_i be the event that every path emanating from L_i satisfies (2.4.8). Since we are working on the strip T_s, we have that $\forall i, j\ \mathbb{P}(E_i) = \mathbb{P}(E_j)$. There are $O(1)$ line-segments and therefore, if E_i holds *whp*, then so does $E = \cap_i E_i$. So we can fix i and consider E_i only. Let $(0, y)$ be the midpoint of L_i. Then by Claim 1 we have

$$y_+ \leq y + s/2 + \varepsilon s/2 \tag{2.4.9}$$

and

$$y_- \geq y - s/2 - \varepsilon s/2. \tag{2.4.10}$$

Since the length of L_i is $\varepsilon s/2$, we have that $|y_0 - y| \leq \varepsilon s/4$ and thus, (2.4.9) implies that the upper bound in (2.4.8) holds *whp*. For the lower bound assume that it does not hold and therefore, the probability of having a black crossing of T_s starting in L_i and satisfying

$$y_+ \le y_0 + s/2 - \varepsilon s \le y + s/2 - 3\varepsilon s/4 \tag{2.4.11}$$

is bounded away from zero along some sequence of s values going to infinity. However, if both (2.4.10) and (2.4.11) hold, then there is a horizontal black crossing of the rectangle $[0,s]\times[y-s/2-\varepsilon s/2, y+s/2-3\varepsilon s/4]$ which happens with probability tending to 0 by (2.4.4). Therefore, since (2.4.10) holds *whp*, the probability of having a black crossing satisfying (2.4.11) must tend to 0 as $s\to\infty$, proving the lower bound in (2.4.8). □

The following claim is the first real suggestion that under assumption (2.4.4) the crossings of certain rectangles have strange properties *whp*.

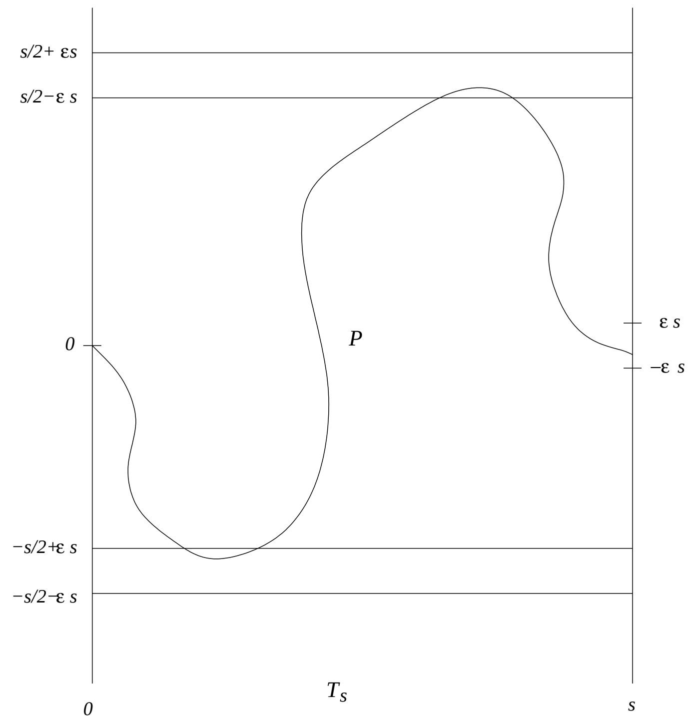

Figure 2.3: According to Claim 2, the crossing of the strip T_s must have very special structure.

Claim 3 ([11], Claim 4.5): *Let $C>0$ fixed, and let $R=R_s$ be the s by $2Cs$ rectangle $[0,s]\times[-Cs,Cs]$. For $i=0,1$, set $R_i=[is/100,(i+99)s/100]\times[-Cs,Cs]$. Assuming that* (2.4.4) *holds,* whp *every black path P crossing R horizontally contains disjoint paths P_0 and P_1 such that P_i crosses R_i horizontally.*

Proof. We cover the left-hand side of R by $O(1)$ line-segments L_i of length $s/1000$. Let P be a horizontal black crossing of R starting, say, at $(0,y_0)$, which falls into L_i for some i, and ending at (s, y_1). We denote by $(0, y)$ the midpoint of L_i. Let P_0 be the initial segment of P up to the point it first crosses the line $x = 0.99s$ and let P_1 be the last segment of P after it crossed the line $x = 0.01s$ the last time. See Figure 2.4 for a picture of P_0 and P_1.

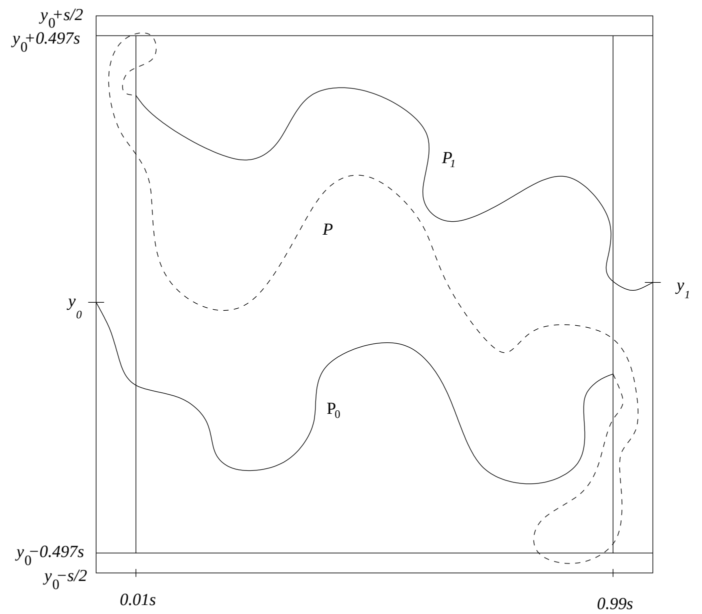

Figure 2.4: The initial and the last segment of a crossing P of the rectangle R_s. Note that if P_0 and P_1 are not disjoint, their union then contains a crossing of an s by $0.994s$ rectangle which has probability tending to zero.

We show that *whp* P_0 and P_1 are disjoint. By Claim 2 they both satisfy

$$\big||y_\pm(P_i) - y_j(P_i)| - 0.495s\big| \le s/1000, \tag{2.4.12}$$

for $i = 0, 1$ and $j = 0, 1$. Applying Claim 2 for P with $\varepsilon = 1/2000$ shows that $|y_0(P) - y_1(P)| \le s/1000$. We also have that $|y - y_0(P)| \le s/2000$ and hence both P_0 and P_1 lie entirely within the rectangle $[0, s] \times [y - 0.497s, y + 0.497s]$. But then they must be disjoint *whp*, otherwise there would be a horizontal crossing of the above mentioned s by $0.994s$ rectangle, which has probability tending to 0 by (2.4.4). Since there are $O(1)$ line-segments, the claim follows. $\square$

The idea of the proof of Claim 3 can be iterated to result in an even less plausible picture for the crossings of rectangles.

Claim 4 ([11], Claim 4.6): *Let $C > 0$ be fixed, and let $R = R_s$ be the s by $2Cs$ rectangle $[0, s] \times [-Cs, Cs]$. For $0 \le j \le 4$, set $R_j = [js/100, (j + 96)s/100] \times [-Cs, Cs]$. Assuming that* (2.4.4) *holds, whp every black path P crossing R horizontally contains* 16 *disjoint black paths P_i, $1 \le i \le 16$, where each P_i crosses some R_j horizontally.*

Proof. We only give a sketch of the proof. Repeating the argument of the proof of Claim 3 shows that any path P_0 obtained as in that proof can be divided into two disjoint paths P_{00} and P_{01} so that they both cross rectangles of width $0.98s$ and height s. We can continue dividing every path till we reach the desired number of paths. □

Following [11] we now define, for a rectangle R, the random variable $L(R)$ as the minimum length of a black path crossing R horizontally. (More precisely, it is the minimum length of a piecewise-linear black curve that crosses R horizontally). If there is no horizontal black crossing of R we take $L(R) = \infty$. So far the dependencies in our model caused no problem but now if R_1 and R_2 are two disjoint rectangles, $L(R_1)$ and $L(R_2)$ are not independent (no matter how large the distance between the two rectangles). Therefore, a suitable modification $\tilde{L}$ is introduced. The key idea is that *whp* the colours inside a rectangle with length and width of order s, are completely determined by the Poisson points within distance of order $o(s)$ of the rectangle. Therefore, we can define an event that holds *whp* and conditioned on this event $L(R_1)$ and $L(R_2)$ are independent if R_1 and R_2 are far enough from each other. Recall that the Poisson process is denoted by Z. For a given rectangle R, the r–neighbourhood of R is the set of all points within distance r of some point of R. The r–neighbourhood of R is denoted by $R[r]$. For an arbitrary ρs by s rectangle R_s, we define $E_{\text{dense}}(R_s)$ to be the event that for every point $x \in R_s[r]$ there is some point $z \in Z$ with $\text{distance}(x, z) < r$, where $r = 2\sqrt{\log s}$. The following lemma contains some important properties of $E_{\text{dense}}(R_s)$.

Lemma 2.4.5. *([11], Lemma 3.2): Let $\rho \geq 0$ be fixed and $R_s \subset \mathbb{R}^2$ be a ρs by s rectangle. Furthermore, let $r = 2\sqrt{\log s}$. Then the event $E_{dense}(R_s)$ holds* whp *and it depends only on the restriction of Z to $R_s[2r]$. Finally, if $E_{dense}(R_s)$ holds, the colour of every point of R_s in the Voronoi model is determined by the restriction of $(Z, \text{colours of } z \in Z)$ to $R_s[2r]$.*

We omit the proof of this lemma, which follows fairly easily from the standard properties of Poisson point processes; see [11] for the details.

We use the event $E_{\text{dense}}(R_s)$ to define $\tilde{L}$. For a given ρs by s rectangle R_s let $\tilde{L}(R_s)$ be equal to $L(R_s)$ if $E_{\text{dense}}(R_s)$ holds and $\tilde{L}(R_s) = 0$ otherwise. Then by Lemma 2.4.5 we have that

$$\tilde{L}(R_s) = L(R_s), \text{ whp.} \tag{2.4.13}$$

Since the length of any horizontal crossing of R_s is at least s, we have that *whp*

$$\tilde{L}(R_s) \geq s. \tag{2.4.14}$$

The sole reason of introducing $\tilde{L}$ is to control the dependencies in the model. The following claim says that if two rectangles, R_1 and R_2, are far enough from each other then $\tilde{L}(R_1)$ and $\tilde{L}(R_2)$ are independent.

Claim 5 ([11], Claim 4.7): *Let R_1 and R_2 be two s by $2s$ rectangles, separated by a distance of at least $s/100$. If s is large enough, then the random variables $\tilde{L}(R_1)$ and $\tilde{L}(R_2)$ are independent.*

Proof. By Lemma 2.4.5, we know that if $E_{\text{dense}}(R_s)$ holds then the colour of every point in R_s is determined by the restriction of $(Z, \text{colours of } z \in Z)$ to $R_s[4\sqrt{\log s}]$. Furthermore, $E_{\text{dense}}(R_s)$ is completely determined by the points of Z in $R_s[4\sqrt{\log s}]$. Combining these two facts yields the claim. □

Remark 2.4.6. *In fact, the independence property of $\tilde{L}$ is only used in the proof of Claim 6, and there it could be replaced by the following property (which follows from* (2.4.13) *and Claim 5):*

For each $\varepsilon > 0$ there is a u such that for all $s > u$ and all s by $2s$ rectangles R_1 and R_2 that are separated by a distance of at least $s/100$, we have

$$\sup_{x,y>0} |P(L(R_1) < x,\, L(R_2) < y) \,-\, P(L(R_1) < x)\, P(L(R_2) < y)| < \varepsilon. \tag{2.4.15}$$

We make this remark because the random variables corresponding to $\tilde{L}$ in the self-destructive percolation settings are not independent but (2.4.15) *still holds.*

Now choose an arbitrary number $\hat{\eta}$ smaller than 10^{-4} and define

$$\hat{t}(s) = \sup\{x \,:\, \mathcal{P}(\tilde{L}(R_s) < x) \le \hat{\eta}\}, \tag{2.4.16}$$

where R_s is an s by $2s$ rectangle. This definition of $\hat{t}$ is the same in form as that of t in [11] (see two lines below (4.15) in [11]); however our way of choosing $\hat{\eta}$ was different. A consequence of this difference is that, in our setup, $\hat{t}(s)$ can be ∞. Note that it immediately follows from (2.4.14) and the definition of $\hat{t}$ that

$$\hat{t}(s) \ge s, \text{ for all sufficiently large } s. \tag{2.4.17}$$

We will show that under (2.4.4) the function $\hat{t}(s)$ has to grow very fast and consequently so does $\tilde{L}(R_s)$, at least on a set with probability arbitrary close to one.

Claim 6 ([11], Claim 4.8): *Let R_s be a fixed $0.96s$ by $2s$ rectangle. If* (2.4.4) *holds, then*

$$\mathcal{P}\left(L(R_s) < \hat{t}(0.47s)\right) \le 200\hat{\eta}^2, \tag{2.4.18}$$

for every large enough s.

Proof. Without loss of generality we can take $R_s = [0, 0.96s] \times [-s, s]$. We cover the left-hand side of R_s by 100 line-segments L_i of length 0.02s. We pick an arbitrary i and first focus only on paths emanating from L_i. Let B_i

denote the event that there is a path P emanating from L_i that crosses R_s and has length shorter than $\hat{t}(0.47s)$. We first show that $\mathbb{P}(B_i) \leq 2\hat{\eta}^2$.

Let $(0, y)$ be the midpoint of L_i and consider a path P starting at $(0, y_0) \in L_i$ and ending at $(0.96s, y_1)$. Let P_0 be the initial segment of P until it first crosses the line $x = 0.47s$ and P_1 be the final segment after it crossed the line $x = 0.49s$ for the last time. Let $R_0 = [0, 0.47s] \times [y - 0.47s, y + 0.47s]$ and $R_1 = [0.49s, 0.96s] \times [y - 0.47s, y + 0.47s]$. See Figure 2.5.

Combining the results of the previous claims we get that the following properties all hold *whp*

1. $|y_0 - y| \leq 0.01s$ since L_i has a total length of $0.02s$.
2. $|y_1 - y_0| \leq 0.01s$ by (2.4.7).
3. P_0 is contained in R_0 by (2.4.6) and 1. above.
4. P_1 is contained in R_1 by (2.4.6) and 1. and 2. above.

Therefore it follows from 3. and 4. above that *whp* every path P starting in L_i has length at least $L(R_0) + L(R_1)$ and therefore also *whp* at least $\tilde{L}(R_0) + \tilde{L}(R_1)$. Furthermore, if s is large enough then $L(R_0)$ and $L(R_1)$ are independent by Claim 5. From the definition of $\hat{t}(s)$ it is immediate that $\mathbb{P}(\tilde{L}(R_s) < \hat{t}(s)) \leq \hat{\eta}$ for every large enough s. Taking $0.47s$ in place of s and combining this with the independence shows that

$$\mathbb{P}\big(\tilde{L}(R_0) + \tilde{L}(R_1) < \hat{t}(0.47s)\big) \leq \hat{\eta}^2. \tag{2.4.19}$$

Thus, the probability that a path P emanating from L_i has length smaller than $\hat{t}(0.47s)$ is at most $\hat{\eta}^2 + o(1) \leq 2\hat{\eta}^2$.

Now first note that $L(R_s) = \infty$ if there is no horizontal black crossing of R_s. Second, note also that we made no use of the exact location of L_i and the argument above is valid for any possible value of i. Then (2.4.18) follows immediately. $\square$

From the results up to now (in particular Claim 4, Claim 6 and (2.4.13)), the following quite startling Proposition follows easily.

Proposition 2.4.7. *If* (2.4.4) *holds, then, for all sufficiently large s,*

$$\hat{t}(s) \geq 16\hat{t}(0.47s). \tag{2.4.20}$$

Proof. Without loss of generality we can consider the s by $2s$ rectangle $R_s = [0, s] \times [-s, s]$ and five sub-rectangles $R_j = [js/100, (j + 96s)/100] \times [-s, s]$, $0 \leq j \leq 4$, of R_s. By Claim 4 *whp* every black horizontal crossing P of R_s contains 16 disjoint black paths P_i so that each P_i crosses some R_j horizontally. Therefore, P has length at least $16 \min_j L(R_j)$. Furthermore, for each rectangle R_j we have from Claim 6 that $\mathbb{P}(L(R_j) \geq \hat{t}(0.47s)) \geq 1 - 1000\hat{\eta}^2$. Thus, combining these last two facts yields that if s is large enough then with probability at least $1 - 1001\hat{\eta}^2$ we have $L(R_s) \geq 16\hat{t}(0.47s)$. Since *whp* $\tilde{L}(R_s) = L(R_s)$, see (2.4.13), we have for every large enough s that

$$\mathbb{P}\big(\tilde{L}(R_s) < 16\hat{t}(0.47s)\big) \leq 1002\hat{\eta}^2 \leq \hat{\eta}, \tag{2.4.21}$$

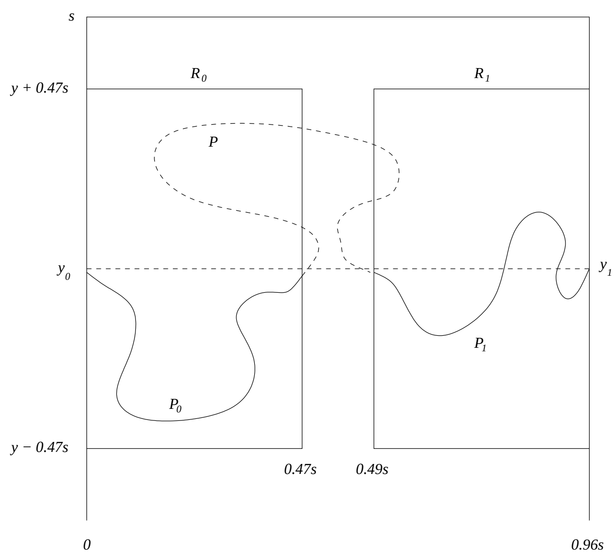

Figure 2.5: *Whp* any crossing of R contains two disjoint paths that cross R_0 and R_1, respectively. Note that the distance between the two sub-rectangles is $2s/100$ and therefore (2.4.15) holds.

where the last inequality follow from our assumption that $\hat{\eta} \leq 10^{-4}$. Now (2.4.20) follows from the definition of $\hat{t}(s)$. □

Corollary 2.4.8. *If* (2.4.4) *holds then for all sufficiently large s we have*

$$\hat{t}(s) > s^3. \tag{2.4.22}$$

Proof. Taking logarithms in (2.4.20) and considering the result in Proposition 2.4.7 shows that for large enough s if $\log s$ increases by $\log(1/0.47)$ then $\log \hat{t}(s)$ increases by at least $\log 16 > 3 \log(1/0.47)$. Since we have that $\hat{t}(s) \geq s$ for large s, it follows that $\hat{t}(s)$ grows at least as fast as some constant times $s^{\log 16/\log(1/0.47)}$, and in particular $\hat{t}(s) \geq s^3$ for every large enough s. □

Proposition 2.4.9. *Let R_s be an s by $2s$ rectangle. Then for every large enough s we have that*

$$\mathcal{P}(s^3 \leq L(R_s) < \infty) \to 0 \text{ as } s \to \infty. \tag{2.4.23}$$

Proof. Let $r = 2\sqrt{\log s}$. The number of points of Z in $R_s[r]$ has Poisson distribution with mean being the area of $R_s[r]$, which is smaller than $(s + 2r)(2s + 2r) < 3s^2$ for every large enough s. So *whp* there are at most $4s^2$ Voronoi points in $R_s[r]$. On the event $E_{\text{dense}}(R_s)$, which holds *whp*, every point in R_s has a Poisson point within distance r. Thus, the intersection of R_s and any Voronoi cell has diameter at most $2r = 4\sqrt{\log s}$. A simple geometric argument shows that with probability 1, each Voronoi cell is a convex polygon and hence the intersection of each Voronoi cell and R_s is a connected set. Therefore, the shortest crossing of R_s intersects each Voronoi cell inside R_s at most once. So *whp* the shortest horizontal crossing of R_s intersects at most $4s^2$ Voronoi cells, each with diameter at most $2r$. Thus, *whp* it cannot have length larger than $8rs^2 < s^3$. Now the Proposition follows since $L(R_s) < \infty$ if and only if there is horizontal black crossing in R_s. $\square$

Now Theorem 2.4.3 follows in a few lines. From Corollary 2.4.8, the definition of $\hat{t}(s)$ and (2.4.13) we get that

$$\limsup_{s\to\infty} \mathcal{P}(L(R_s) < s^3) \leq \hat{\eta}, \tag{2.4.24}$$

where R_s is an s by $2s$ rectangle. Combining (2.4.23) and (2.4.24), and recalling that $L(R_s) = \infty$ iff there is no horizontal black crossing of R_s, immediately gives

$$\limsup_{s\to\infty} \mathcal{P}(\exists \text{ a horizontal black crossing of } R_s) \leq \hat{\eta}.$$

Now, since $\hat{\eta}$ was an arbitrary number between 0 and 10^{-4}, we get $\lim_{s\to\infty} \mathcal{P}(\exists \text{ horizontal black crossing of } R_s) = 0$, that is,

$$f(1/2, s) \to 0, \text{ as } s \to \infty. \tag{2.4.25}$$

Note that in the last part of the above arguments (after Claim 4) we worked in particular with s by $2s$ rectangles. A careful look at the arguments shows that the choice of this factor 2 is, in fact, immaterial: if we would take s by $3s$ rectangles or, more generally, fix an $N \geq 2$ and take s by Ns rectangles, the arguments remain practically the same. To see this, one can easily check that in Claims 1 - 4 the factor 2 plays no role at all: here the rectangles under consideration are s by $2Cs$, where C is a fixed but arbitrary positive number. Further, the proof of Claim 5 remains the same when, for some fixed positive number C, we replace the factor 2 by $2C$. And, the definition of $\hat{t}(s)$ (see (2.4.16)), which was given in terms of s by $2s$ rectangles, has, for each $C > 0$, an obvious analog for s by $2Cs$ rectangles:

$$t_C(s) := \sup\{x : \mathcal{P}(\tilde{L}(R_s^C) < x) \leq \hat{\eta}\}, \tag{2.4.26}$$

where, for each s, R_s^C is some fixed s by $2Cs$ rectangle.

In the generalization of Claim 6 we now fix $C \geq 1$, and take $R_s^C := [0, 0.96s] \times [-Cs, Cs]$. In the proof of this Claim we have to replace, on the vertical scale, s by Cs. For instance, the segments L_i, which in the original proof in [11] have length $0.02s$, will now have length $0.02Cs$, and R_0 and R_1 which in the original proof are $0.47s$ by $2 \times 0.47s$ rectangles, are now $0.47s$ by $2C0.47s$ rectangles. In this way we get if (2.4.4) holds, for each fixed $C > 1$ the following analog of (2.4.25):

$$f(1/(2C), s) \to 0, \text{ as } s \to \infty. \tag{2.4.27}$$

This proves Theorem 2.4.3 and hence Theorem 2.4.2.

□

In the above we were dealing with black horizontal crossings. Obviously, completely analogous results hold for white horizontal crossings: If we denote (for a fixed value of the parameter p of the Voronoi percolation model), the probability of a vertical white crossing of a given ρs by s rectangle by $g(\rho, s)$, we have that if $\lim_{s\to\infty} g(\rho, s) = 0$ for *some* $\rho > 0$, then this limit is 0 for *all* $\rho > 0$. Since a rectangle has either a horizontal black crossing or a vertical white crossing (and hence $g(\rho, s) = 1 - f(\rho, s)$) this gives:

Corollary 2.4.10.

$$\begin{aligned} &\textit{If} \qquad \lim_{s\to\infty} f(\rho, s) = 1 \textit{ for some } \rho > 0, \\ &\textit{then} \qquad \lim_{s\to\infty} f(\rho, s) = 1 \textit{ for all } \rho > 0, \end{aligned} \tag{2.4.28}$$

2.4.3 An RSW analog for self-destructive percolation

In the previous subsection we considered (and somewhat strengthened) an RSW-like result of Bollobás and Riordan ([11]) for the Voronoi percolation model. Only a few properties of the model are used in its proof. As remarked in [11] (at the end of Section 4; see also [13], Section 5.1), these properties are basically the following: First of all, crossings of rectangles are defined in terms of 'geometric paths' in such a way that, for example, horizontal and vertical black crossings meet, which enables to form longer paths by pasting together several small paths. Further, a form of FKG is used (e.g. that events of the form 'there is a black path from A to B' are positively correlated. Also some symmetry is needed. Bollobás and Riordan say that "invariance of the model under the symmetries of $\mathbb{Z}^2$ suffices, as we need only consider rectangles with integer coordinates". Finally, some form of asymptotic independence is needed (see Remark 2.4.6). Similar considerations hold with respect to the somewhat stronger Theorem 2.4.2.
Using the results in Section 2.2, is not difficult to see that the SDP model has the above mentioned properties:

- The indicated geometric properties are just the well-known intersection properties of paths in the square lattice (and in its matching lattice).
- Lemma 2.2.1 gives the needed FKG-like properties.
- Asymptotic independence: Note that for $p \le p_c$ the SDP model is an ordinary percolation model, where this property is trivial. If $p > p_c$, then $1 - p$ is smaller than the critical probability of the matching lattice. In

that case the needed asymptotic independence (of the form described in Remark 4.6) comes from Lemma 2.2.3 and the well-known exponential decay theorems for ordinary subcritical percolation.

- The SDP model on the square lattice clearly has all the symmetries of $\mathbb{Z}^2$.

Further, to carry out for the SDP model the analog of the arguments that led from Theorem 2.4.3 to Corollary 2.4.10, we note that the random collection of vacant sites on the matching lattice clearly also has the above mentioned properties. So we get the following theorem for the SDP model:

Theorem 2.4.11. *The analogs of Theorems 2.4.2 and 2.4.3 and Corollary 2.4.10 hold for the self-destructive percolation model. In particular, let for the SDP model with parameters p and δ, $f(\rho, s) = f_{p,\delta}(\rho, s)$ denote the probability that there is an occupied horizontal crossing of a given $\rho s \times s$ rectangle. We have*

$$\begin{aligned} &\textit{If} \qquad \lim_{s\to\infty} f(\rho, s) = 1 \textit{ for some } \rho > 0, \\ &\textit{then} \qquad \lim_{s\to\infty} f(\rho, s) = 1 \textit{ for all } \rho > 0. \end{aligned} \tag{2.4.29}$$

In the next sections this result will play an important role in the completion of the proof of Theorem 2.1.9. In particular, in Section 2.5 it will be used to prove a finite-size criterion for supercriticality of the SDP model.

2.5 A finite-size criterion

The main result of this section is a suitable finite-size criterion for supercriticality of the SDP model. The overall structure of the argument is similar to that in ordinary percolation (see [19]), but the dependencies in the model require extra attention. One of the main ingredients, a suitable RSW-like theorem for this model, was obtained in the previous section.

Theorem 2.5.1. *Let $f = f_{p,\delta}$ as in Theorem 2.4.11. There is a universal constant $\alpha > 0$ and there is a decreasing function $\hat{N} : (p_c, 1) \to \mathbb{N}$ such that for all $p > p_c$ and all $\delta > 0$ the following two assertions, (i) and (ii) below, are equivalent.*

$$\begin{aligned} &i. \;\; \theta(p, \delta) > 0. \\ &ii. \exists n \geq \hat{N}(p) \textit{ such that } f_{p,\delta}(3, n) > 1 - \alpha. \end{aligned} \tag{2.5.1}$$

Remark 2.5.2. *In ordinary percolation $\hat{N}(p)$ can be taken constant 1. Remark 2.6.1 below explains the impact of this difference.*

Proof. Consider for each $n \in \mathbb{N}$ the events

$$A = \{\exists \text{ a vertical vacant *-crossing of } [0,9n] \times [0,3n]\}; \quad (2.5.2)$$
$$B = \{\exists \text{ a vertical vacant *-crossing of } [0,9n] \times [0,n]\};$$
$$C = \{\exists \text{ a vertical vacant *-crossing of } [0,9n] \times [2n,3n]\}.$$

Let $h(\rho, n)$ denote the probability of a vertical vacant *-crossing (in the matching lattice) of a ρn by n box. So, $h(\rho, n) = 1 - f(\rho, n)$. Clearly, $\mathcal{P}_{p,\delta}(B) = \mathcal{P}_{p,\delta}(C) = h_{p,\delta}(9, n)$ and $\mathcal{P}_{p,\delta}(A) = h_{p,\delta}(3, 3n)$. It is also clear that $A \subset B \cap C$. From this, Lemma 2.2.3, the fact that the r.h.s. of (2.2.1) is decreasing, and the well-known exponential decay results for ordinary subcritical percolation applied to (2.2.1), it follows that there is an increasing, function $\phi : (p_c, 1) \to (0, \infty)$ such that for all $p > p_c$,

$$h(3, 3n) \leq h(9, n)^2 + \exp(-n\phi(p)). \quad (2.5.3)$$

Further note that if the event B occurs, there must be a vacant vertical *-crossing of one of the rectangles $[0,3n] \times [0,n]$, $[2n,5n] \times [0,n]$, $[4n,7n] \times [0,n]$, $[6n,9n] \times [0,n]$, or a vacant horizontal *-crossing of one of the rectangles $[2n,3n] \times [0,n]$, $[4n,5n] \times [0,n]$, $[6n,7n] \times [0,n]$.

Hence

$$h(9, n) \leq 4h(3, n) + 3h(1, n) \leq 7h(3, n), \quad (2.5.4)$$

which combined with (2.5.3) gives

$$h(3, 3n) \leq 49h(3, n)^2 + \exp(-n\phi(p)). \quad (2.5.5)$$

Take α so small that $49\alpha^2 < \alpha/4$. Let, for each $p > p_c$, $\hat{N}(p)$ be the smallest positive integer for which

$$\exp(-\hat{N}(p)\phi(p)) < \alpha/4.$$

Sine ϕ is increasing, $\hat{N}$ is decreasing in p.

Now suppose $p > p_c$ and $\delta \in (0, 1)$ are given and suppose that (ii) holds. So there exists an n that satisfies:

$$\exp(-n\phi(p)) < \alpha/4 \text{ and } h(3, n) < \alpha. \quad (2.5.6)$$

From (2.5.5), (2.5.6) and the choice of α we get

$$h(3, 3n) \leq 49\alpha^2 + \alpha/4 < \alpha/4 + \alpha/4 = \alpha/2, \quad (2.5.7)$$

and

$$\exp(-3n\phi(p)) < (\alpha/4)^3 < (\alpha/2)/4.$$

Hence, (2.5.6) with n replaced by $3n$, and α replaced by $\alpha/2$ holds. So we can iterate (2.5.7) and conclude that, for all integers $k \geq 0$,
$h(3,\, 3^k n) < \alpha/(2^k)$.

The last part of the argument is exactly as for ordinary percolation: Note that if none of the rectangles $[0, 3^{2k+1}n] \times [0, 3^{2k}n]$ and $[0, 3^{2k+1}n] \times [0, 3^{2k+2}n]$, $k = 0, 1, 2, \cdots$ has a white *-crossing in the 'easy' (short) direction, then each of these rectangles has a black crossing in the long direction. Moreover, all these black crossings together form an infinite occupied path. Hence,

$$\theta(p, \delta) \geq 1 - \sum_{k=0}^{\infty} h(3, n3^k) \geq 1 - \alpha \sum_{k=0}^{\infty} (1/2)^k = 1 - 2\alpha > 0.$$

This proves that (ii) implies (i).

Now we show that (i) implies (ii): Suppose $\theta(p, \delta) > 0$. Then there is (a.s.) an infinite occupied cluster, and by Lemma 2.2.4 this cluster is unique. From the usual spatial symmetries, positive association, and the above mentioned uniqueness one can, in exactly the same way as for ordinary percolation (see [26], Theorem 8.97) show that $f(1, n) \to 1$ as $n \to \infty$. By Theorem 2.4.11 it follows that also $f(3, n) \to 1$ as $n \to \infty$; so (ii) holds. $\square$

2.6 Proof of Theorem 2.1.9

We are now ready to prove Theorem 2.1.9:

Proof. For $p < p_c$, we have (see Section 1) $\theta(p, \delta) = \theta(p + (1 - p)\delta)$, so that continuity follows from continuity for ordinary percolation. If $p = p_c$ and $\delta > p_c + \varepsilon$ for some $\varepsilon > 0$, then (trivially) there is a neighborhood of (p, δ) where the SDP model dominates ordinary percolation with parameter $p_c + \varepsilon/2 > p_c$; hence $\theta(\cdot, \cdot) > 0$ on this neighborhood, and Proposition 2.3.3 implies continuity of $\theta(\cdot, \cdot)$ at (p_c, δ). (In fact, by combining this argument with an Aizenman-Grimmett type argument (see [3]), one can extend this result and show that there is an $\varepsilon > 0$ such that $\theta(\cdot, \cdot)$ is continuous at (p_c, δ) if $\delta > p_c - \varepsilon$).

Finally, we consider the case where $p > p_c$. If $\theta(p, \delta) = 0$, continuity at (p, δ) follows from part (b) of Proposition 2.3.3. So suppose $\theta(p, \delta) > 0$. Let α as in Theorem 2.5.1. By that theorem there is an $n \geq \hat{N}(p)$ with

$$f_{p,\delta}(3, n) > 1 - \alpha.$$

Hence, by Lemma 2.3.1 there is an open neighborhood W of (p, δ) such that

$$f_{p',\delta'}(3, n) > 1 - \alpha, \tag{2.6.1}$$

for all $(p', \delta') \in W$. Since $n \geq \hat{N}(p)$ and $\hat{N}(.)$ is decreasing, it follows from (2.6.1) and Theorem 2.5.1 that $\theta(\cdot, \cdot) > 0$ on S, where S is the set of all

$(p', \delta') \in W$ with $p' \geq p$. From this and Lemma 2.2.2 we conclude that $\theta(\cdot, \cdot)$ is also strictly positive on the set

$$U := \{(p', \delta') \,:\, p' < r \text{ and } p' + (1 - p')\delta' > r + (1 - r)\beta \text{ for some } (r, \beta) \in S\}.$$

It is easy to see that $S \cup U$ contains an open neighborhood of (p, δ). Now it follows from part (a) of Proposition 2.3.3 that $\theta(\cdot)$ is continuous at (p, δ). This completes the proof of the main theorem. □

Remark 2.6.1. *A crucial role in the proof is the finite-size criterion, Theorem 2.5.1. That theorem has been formulated for $p > p_c$. When $p = p_c$ (or $< p_c$) the SDP model is an ordinary percolation model, for which a similar criterion is known. In fact, for ordinary percolation we do not have the dependency problems which led to the introduction of $\hat{N}$. Consequently, for $p = p_c$ we can take $\hat{N} = 1$. But, on the other hand, if we let p tend to p_c from above, the upper bound on $\hat{N}(p)$ obtained from our arguments in Section 2.5 tends to ∞. And that, in turn, comes from the fact that our bound on dependencies, Lemma 2.2.3, is in terms of path probabilities for an ordinary percolation model (on the matching lattice, with parameter $1 - p$) which is subcritical but approaches criticality (which makes these bounds worse and worse) as p approaches p_c from above. This is essentially why the proof of Theorem 2.1.9 does not work at p_c. Of course, if it would work, the conjecture referred to in Section 1 would be false. We hope that attempts to stretch the arguments in our paper as far as possible will substantially increase insight in the conjecture and help to obtain a solution.*

Chapter 3

Invasion Percolation

3.1 Introduction

As we mentioned in the introductory Chapter 1, self-organized criticality has become a great interest in recent years. Although there is no general definition for it, we can say that a system or model has this property if the definition of the model requires no parameter, yet some characteristics of the model resemble those at criticality of a parametric model with a phase transition. One such model is invasion percolation, a stochastic growth model that mirrors aspects of the critical Bernoulli percolation picture without tuning any parameter. The invasion model was introduced independently by two groups ([18] and [36]), who studied it numerically. The first mathematically rigorous study of invasion percolation appeared in [21]. Connections between the invasion cluster and critical Bernoulli percolation have been established for instance in [21], [30], [52], and [53], using both heuristics and rigorous arguments. These results indicated so many parallels between the invaded region and the incipient infinite cluster that a natural question arose: to what extent are these objects similar? This question was studied on the regular tree in [4]. It was shown that although the invaded region and the incipient infinite cluster are locally similar, globally they differ significantly. In this chapter, we prove local similarities between critical Bernoulli clusters and certain invaded clusters (the ponds) in the plane. We also show that globally the invaded region and the incipient infinite cluster are essentially different.

In the remainder of this section we define the invasion percolation model and, using results of [21], we introduce the ponds of the invasion. Then we review results concerning relations between invasion percolation and critical Bernoulli percolation. Finally, we state the main results of this chapter.

3.1.1 The model

For simplicity *we restrict ourselves here to the square lattice*. Invasion percolation can be similarly defined on other two dimensional lattices, and the results of this chapter still hold for lattices which are invariant under reflection in one of the coordinate axes and under rotation around the origin by some angle in $(0,\pi)$. In particular, this includes the triangular and honeycomb lattices.

Although our results concern invasion in the plane, we give the definition of invasion percolation for $\mathbb{Z}^d$. Consider the hypercubic lattice $\mathbb{Z}^d$ with its set of nearest neighbour bonds $\mathbb{E}^d$. We denote edges by their endpoints, i.e. we write $e = \langle x, y\rangle$ if the two endpoints of e are x and y. Let $G = (V, E)$ be an arbitrary subgraph of $(\mathbb{Z}^d, \mathbb{E}^d)$ and we define the outer edge boundary, ΔG of G as

$$\Delta G = \{e = \langle x, y\rangle \in \mathbb{E}^d \ : \ e \notin E(G), \text{ but } x \in G \text{ or } y \in G\}.$$

The first step is to assign independent random variables, uniformly distributed in $[0,1]$, to each bond $e \in \mathbb{E}^d$. We denote these variables by τ_e. Using them, we recursively define an increasing sequence $G_0, G_1, G_2, \ldots$ of connected subgraphs of the lattice. G_0 only contains the origin, with no edges. Once $G_i = (V_i, E_i)$ is defined, we select the edge e_{i+1} that minimizes τ on ΔG_i. We take $E_{i+1} = E_i \cup \{e_{i+1}\}$ and let G_{i+1} be the graph induced by the edge set E_{i+1}. The graph G_i is called the invaded region at time i, and the graph $\mathcal{S} = \cup_{i=0}^{\infty} G_i$ is called the *invasion percolation cluster* (IPC). Let $E_\infty = \cup_{i=0}^{\infty} E_i$.

Since we would like to compare Bernoulli percolation to the invasion, we use a well-known analogous definition of Bernoulli percolation that makes the coupling of the two models immediate. For any $p \in [0,1]$ we say that an edge $e \in \mathbb{E}^d$ is p-open if $\tau_e < p$. It is obvious that the resulting random graph of p-open edges has the same distribution as the one obtained by declaring each edge of $\mathbb{E}^d$ open with probability p and closed with probability $1-p$, independently of the states of all other edges. The percolation probability $\theta(p)$ is the probability that the origin is in the infinite cluster of p-open edges. There is a critical probability $p_c = \inf\{p \ : \ \theta(p) > 0\} \in (0,1)$. For general background on Bernoulli percolation we refer the reader to [26].

One can show that for all $p > p_c$ the invasion intersects the infinite p-open cluster with probability one. This was first proved in [21] assuming a famous conjecture of the time, concerning bond percolation in slabs, holds. That conjecture was later proved in [27]. In case $d = 2$ this result immediately follows from the Russo-Seymour-Welsh theorem (Appendix A). Furthermore, the definition of the invasion mechanism implies that if the invasion reaches the p-open infinite cluster for some p, it will never leave this cluster. Combining these facts yields that if e_i is the edge added at time i then $\limsup_{i\to\infty} \tau_{e_i} = p_c$. From now on, we consider only $d = 2$. In this case, it is well-known that $\theta(p_c) = 0$, which implies that for every $t > 0$ there is an edge $e(t)$ such that $e(t)$ is invaded after step t and $\tau_{e(t)} > p_c$. The last two results give that $\hat{\tau}_1 = \max\{\tau_e \ : \ e \in E_\infty\}$ exists and is greater than p_c. Let $\hat{e}_1$ denote the edge at which the maximum value of τ is taken and assume that $\hat{e}_1$ is invaded at step $i_1 + 1$. Following the terminology of [41], we call the graph G_{i_1} the *first pond* of the invasion, and we denote it $\hat{V}_1$. The edge $\hat{e}_1$ is called the *first outlet*. The second pond of the invasion is defined similarly. Note that the same argument as above implies that $\hat{\tau}_2 = \max\{\tau_{e_i} \ : \ e_i \in E_\infty, i > i_1\}$ exists and is greater than p_c. If we assume that $\hat{\tau}_2$ is taken on the edge $\hat{e}_2$ at step $i_2 + 1$, we call the graph $G_{i_2} \setminus G_{i_1}$ the *second pond* of the invasion, and we denote it $\hat{V}_2$. The further ponds $\hat{V}_k$ can be defined analogously.

The following interpretation gives a natural meaning to the ponds. Consider an infinite piece of land divided into square parcels. These parcels are separated by dikes whose heights are given by the values of independent random variables, uniformly distributed on $[0, 1]$. One of the parcels, called the parcel of the origin, contains an infinite source of water. First, the water level in the parcel of the origin rises until it reaches the height of the lowest adjacent dike and then it spills over into the parcel on the other side of this dike. Next, the water level rises in both parcels until it reaches the height of the lowest dike on the boundary of the union of the two parcels, at which time a new parcel floods. The process continues indefinitely, and as time approaches infinity, an infinite region of land will flood. Consider the dual lattice of $\mathbb{Z}^2$, each dual edge having the τ value of its corresponding edge in the original lattice, identifying the dual edges with the dikes and the origin with the source of water. Each vertex of $\mathbb{Z}^2$ corresponds to exactly one parcel of land. It is evident from the invasion mechanism and from the way the flood spreads on the land that a parcel is flooded if and only if the corresponding vertex of $\mathbb{Z}^2$ is invaded. Now we explain the meaning of the first pond in the flood setting. At step i_1, when the first outlet is invaded, the minimal τ value on the boundary of G_{i_1} is that of $\hat{e}_1$. However this is the edge with the largest τ value ever added to the invasion. This means that the invasion will never return to G_{i_1}, i.e. no edge on ΔG_{i_1} other than $\hat{e}_1$ will be invaded. Therefore, after some time, all water will flow over the dike corresponding to $\hat{e}_1$ and the water level in each parcel of the first pond will be constant and equal to $\hat{\tau}_1$. The same argument shows that after some time, the water level in the second pond will become and remain $\hat{\tau}_2$, and so on.

Now that our model is defined we review a few results that established connections between the invasion and the critical percolation models. To our best knowledge, the first paper with mathematically rigorous results in this area is [21], where it is shown, among other things, that the empirical distribution of the τ value of the invaded edges converges to the uniform distribution on $[0, p_c]$. Results on the fractal nature of the invaded region are also obtained in [21]. The authors showed that the region has zero volume fraction, given that there is no percolation at criticality, and that it has boundary to volume ratio $(1 - p_c)/p_c$. This corresponds to the asymptotic boundary to volume ratio for large critical clusters (see [40] and [35]). The above results indicate that a large proportion of the edges in the IPC belongs to big p_c-open clusters.

An object that turns out to be closely related to the invaded region is the incipient infinite cluster (IIC). Loosely speaking, one can say that the IIC is the “infinite open cluster at criticality”. The IIC can be constructed by conditioning on the origin being connected to a site at distance n from the origin in critical percolation and by considering the cluster of the origin. If we let $n \to \infty$, an infinite cluster is obtained, and this cluster is called the incipient infinite cluster. (Later in this chapter we will give the precise definition. For detailed results on the IIC we refer the reader to [33].) Let S_n be the number of invaded sites within distance at most n from the origin. The scaling of the moments of S_n as n goes to infinity was obtained in [30] and [53] and turned out to coincide with the scaling of the corresponding moments for the IIC. Another similarity established in [30] is concerned with the invasion picture far away from the origin: the invasion measure was shown to be locally the same as the IIC measure. To our best knowledge, the only paper to date concerned with differences between the invasion model and critical percolation is [4]. The authors consider invasion percolation on regular trees. The scaling behaviour of the r-point function and the volume of the invaded region at and below a given height can be explicitly computed. It is found that while the power laws of the scaling are the same for the invaded region and for the incipient infinite cluster, the scaling functions differ, and consequently the two clusters behave differently. In fact, their laws are found to be mutually singular. Even though the arguments of [4] do not work for invasion in the plane, their results give a strong indication that in spite of the presence of many similarities, the two objects are indeed different.

In this chapter we compare connectivity properties of the origin's invaded region to those of the critical percolation cluster of the origin and the IIC. In Theorem 3.1.1 and Theorem 3.1.3 we give the asymptotic behaviour for the k-point function of the first pond. We continue to study the relation between the IPC and large p_c-open clusters in Theorem 3.1.4 and Theorem 3.1.5. We show that, for any K and N, there are infinitely many ponds that contain at least K disjoint p_c-open clusters of size at least N. We also show that given the radius of the first pond is larger than N, the first pond contains at least K disjoint p_c-open clusters of size at least N with probability bounded from below by a positive constant independent of N. For $k \geq 1$, we compute the exact decay rates of the distribution of the radius and the volume of the k-th pond in Theorem 3.1.6 and Theorem 3.1.10. The decay rates of the radius and the volume of the first pond coincide respectively with the decay rates of the distribution of the radius and the volume of the critical cluster of the origin in Bernoulli percolation. However, these decay rates are strictly different from that of the radius and volume of the critical cluster of the origin in case of the kth pond with $k \geq 2$. Finally, in Theorem 3.1.12 we show that the IPC measure and the IIC measure are mutually singular.

3.1.2 Notation

In this section we collect most of the notation and the definitions used in this chapter.

For $a \in \mathbb{R}$, we write $|a|$ for the absolute value of a, and, for a site $x = (x_1, x_2) \in \mathbb{Z}^2$, we write $|x|$ for $\max(|x_1|, |x_2|)$. For $n > 0$ and $x \in \mathbb{Z}^2$, let $B(x,n) = \{y \in \mathbb{Z}^2 \ : \ |y - x| \le n\}$ and $\partial B(x,n) = \{y \in \mathbb{Z}^2 \ : \ |y - x| = n\}$. We write $B(n)$ for $B(0,n)$ and $\partial B(n)$ for $\partial B(0,n)$. For $m < n$ and $x \in \mathbb{Z}^2$, we define the annulus $Ann(x;m,n) = B(x,n) \setminus B(x,m)$. We write $Ann(m,n)$ for $Ann(0;m,n)$.

We consider the square lattice $(\mathbb{Z}^2, \mathbb{E}^2)$, where $\mathbb{E}^2 = \{(x,y) \in \mathbb{Z}^2 \times \mathbb{Z}^2 \ : \ |x - y| = 1\}$. Let $(\mathbb{Z}^2)^* = (1/2, 1/2) + \mathbb{Z}^2$ and $(\mathbb{E}^2)^* = (1/2, 1/2) + \mathbb{E}^2$ be the vertices and the edges of the dual lattice. For $x \in \mathbb{Z}^2$, we write x^* for $x + (1/2, 1/2)$. For an edge $e \in \mathbb{E}^2$ we denote its ends (left respectively right or bottom respectively top) by $e_x, e_y \in \mathbb{Z}^2$. The edge $e^* = (e_x + (1/2, 1/2), e_y - (1/2, 1/2))$ is called the dual edge to e. Its ends (bottom respectively top or left respectively right) are denoted by e^*_x and e^*_y. Note that, in general, e^*_x and e^*_y are not the same as $(e_x)^*$ and $(e_y)^*$. For a subset $\mathcal{K} \subset \mathbb{Z}^2$, let $\mathcal{K}^* = (1/2, 1/2) + \mathcal{K}$. We say that an edge $e \in \mathbb{E}^2$ is in $\mathcal{K} \subset \mathbb{Z}^2$ if both its ends are in $\mathcal{K}$.

Let $(\tau_e)_{e \in \mathbb{E}^2}$ be independent random variables, uniformly distributed on $[0,1]$, indexed by edges. We call τ_e the weight of an edge e. We define the weight of an edge e^* as $\tau_{e^*} = \tau_e$. We denote the underlying probability measure by $\mathbb{P}$ and the space of configurations by $([0,1]^{\mathbb{E}^2}, \mathcal{F})$, where $\mathcal{F}$ is a natural σ-field on $[0,1]^{\mathbb{E}^2}$. We say that an edge e is *p-open* if $\tau_e < p$ and *p-closed* if $\tau_e \ge p$. An edge e^* is *p-open* if e is p-open, and it is *p-closed* if e is p-closed. The event that two sets of sites $\mathcal{K}_1, \mathcal{K}_2 \subset \mathbb{Z}^2$ are connected by a p-open path is denoted by $\mathcal{K}_1 \stackrel{p}{\longleftrightarrow} \mathcal{K}_2$, and the event that two sets of sites $\mathcal{K}^*_1, \mathcal{K}^*_2 \subset (\mathbb{Z}^2)^*$ are connected by a p-closed path in the dual lattice is denoted by $\mathcal{K}^*_1 \stackrel{p^*}{\longleftrightarrow} \mathcal{K}^*_2$.

For positive integers $m < n$, k and $p \in [0,1]$, let $A_{n,p}$ be the event that there is a p-open circuit around the origin of diameter at least n, and let $B_{n,p}$ be the event that there is a p-closed circuit around the origin in the dual lattice of diameter at least n. Let $A_{m,n,p}$ be the event that there is a p-open circuit around the origin in the annulus $Ann(m,n)$, and let $B_{m,n,p}$ be the event that there is a p-closed circuit around the origin in the annulus $Ann(m,n)^*$. Let $A^k_{m,n,p}$ be the event that there are k disjoint p-open paths connecting $B(m)$ to $\partial B(n)$.

For $p \in [0,1]$, we consider a probability space $(\Omega_p, \mathcal{F}_p, \mathbb{P}_p)$, where $\Omega_p = \{0,1\}^{\mathbb{E}^2}$, $\mathcal{F}_p$ is the σ-field generated by the finite-dimensional cylinders of Ω_p, and $\mathbb{P}_p$ is a product measure on $(\Omega_p, \mathcal{F}_p)$, $\mathbb{P}_p = \prod_{e \in \mathbb{E}^2} \mu_e$, where μ_e is given by $\mu_e(\omega_e = 1) = 1 - \mu_e(\omega_e = 0) = p$, for vectors $(\omega_e)_{e \in \mathbb{E}^2} \in \Omega_p$. We say that an edge e is *open* or *occupied* if $\omega_e = 1$, and e is *closed* or *vacant* if $\omega_e = 0$. We say that an edge e^* is *open* or *occupied* if e is open, and it is *closed* or *vacant* if e is closed. The event that two sets of sites $\mathcal{K}_1, \mathcal{K}_2 \subset \mathbb{Z}^2$ are connected by an open path is denoted by $\mathcal{K}_1 \leftrightarrow \mathcal{K}_2$, and the event that two sets of sites $\mathcal{K}^*_1, \mathcal{K}^*_2 \subset \mathbb{Z}^2$ are connected by a closed path in the dual lattice is denoted by $\mathcal{K}^*_1 \stackrel{*}{\leftrightarrow} \mathcal{K}^*_2$.

For positive integers $m < n$ and k, let A_n be the event that there is an occupied circuit around the origin of diameter at least n, and let B_n be the event that there is a vacant circuit around the origin in the dual lattice of diameter at least n. Let $A_{m,n}$ be the event that there is an occupied circuit around the origin in the annulus $Ann(m, n)$, and let $B_{m,n}$ be the event that there is an vacant circuit around the origin in the annulus $Ann(m, n)^*$. Let $A^k_{m,n}$ be the event that there are k disjoint occupied paths connecting $B(m)$ to $\partial B(n)$.

For two functions g and h from a set $\mathcal{X}$ to $\mathbb{R}$, we write $g(z) \asymp h(z)$ to indicate that $g(z)/h(z)$ is bounded away from 0 an ∞, uniformly in $z \in \mathcal{X}$. Throughout this chapter we write log for $\log_2$. We also write $\mathbb{P}_{cr}$ for $\mathbb{P}_{p_c}$. All the constants (C_i) in the proofs are strictly positive and finite. Their exact values may be different from proof to proof.

Most of the constants in this chapter are denoted by C_i. To avoid using large indices we restart the counting in each proof. Since the exact value of these constants is of little importance this should lead to no confusion. Nevertheless, it is important to note that the value of C_i for a fixed i may change between different appearances and the same constant in different proofs may have different index. There are a few constant whose value we would like to keep fixed. Those constants are denoted by D_i.

3.1.3 Main results

Probability for k points in the first pond

Theorem 3.1.1. *Let $\mathcal{C}(0)$ be the cluster of the origin in Bernoulli bond percolation. For any $k > 0$,*

$$\mathbb{P}(x_1, \dots, x_k \in \hat{V}_1) \asymp \mathbb{P}_{cr}(x_1, \dots, x_k \in \mathcal{C}(0)), \qquad x_1, \dots, x_k \in \mathbb{Z}^2. \tag{3.1.1}$$

Remark 3.1.2. *The lower bound follows from the observation that the p_c-open cluster of the origin is a subset of $\hat{V}_1$.*

The reader may ask the question if there is a universal constant c such that, for all $k \geq 1$ and $x_1, ..., x_k \in \mathbb{Z}^2$,

$$\mathbb{P}(x_1, ..., x_k \in \hat{V}_1) \leq c\mathbb{P}_{cr}(x_1, ..., x_k \in \mathcal{C}(0)).$$

In the next theorem we show that the answer to the above question is negative.

Theorem 3.1.3.

$$\lim_{n\to\infty} \frac{\mathbb{P}(B(n) \subset \hat{V}_1)}{\mathbb{P}_{cr}(B(n) \subset \mathcal{C}(0))} = \infty.$$

Ponds and p_c-open clusters

Now we state two theorems which say that invasion ponds can contain several large p_c-open clusters. Let $K \geq 2, N \geq 1$, and let $\mathcal{U}(m, K, N)$ be the event that the m^{th} pond contains at least K disjoint p_c-open clusters of size at least N.

Theorem 3.1.4. *With probability one, there exist infinitely many values of m for which $\mathcal{U}(m, K, N)$ holds.*

Theorem 3.1.5. *There exists $\varepsilon > 0$ independent of N but dependent on K such that*

$$\mathbb{P}(\mathcal{U}(1, K, N)|\hat{R}_1 \geq N) \geq \varepsilon,$$

where $\hat{R}_1$ is the radius of the first pond.

Radii and volumes of the ponds

We define $\hat{R}_j$ to be the radius of the graph G_{i_j}, that is $\hat{R}_j = \max\{|x| \,:\, x \in G_{i_j}\}$. We refer the reader to Section 3.1.1 for the definitions of i_j and G_{i_j}. In the next theorem we give the asymptotics for the radii $\hat{R}_j$.

Theorem 3.1.6. *For any $k \geq 1$,*

$$\mathbb{P}(\hat{R}_k \geq n) \asymp (\log n)^{k-1}\mathbb{P}_{cr}(0 \leftrightarrow \partial B(n)). \tag{3.1.2}$$

Remark 3.1.7. *Let $\{0 \leftrightarrow_k \partial B(n)\}$ be the event that there is a path connecting the origin to the boundary of $B(n)$ such that at most k of its edges are closed. If this event holds we say that the origin is connected to $\partial B(n)$ by an open path with k defects. It is a consequence of the RSW-Theorem (see [44, Proposition 18]) that*

$$\mathbb{P}_{cr}(0 \leftrightarrow_k \partial B(n)) \asymp (\log n)^k \mathbb{P}_{cr}(0 \leftrightarrow \partial B(n)).$$

Therefore, Theorem 3.1.6 implies that, for any $k \geq 1$,

$$\mathbb{P}(\hat{R}_k \geq n) \asymp \mathbb{P}_{cr}(0 \leftrightarrow_{k-1} \partial B(n)).$$

Remark 3.1.8. *Note that in case $k = 1$ the lower bound immediately follows from the fact that $\mathcal{C}(0) \subset \hat{V}_1$, where $\mathcal{C}(0)$ is the p_c-open cluster of the origin for Bernoulli bond percolation. However, in case $k \geq 2$ the lower bound is not trivial.*

Let $\bar{R}_k$ be the diameter of the kth pond, $\bar{R}_k = \max\{|x-y| \,:\, x, y \in \hat{V}_k\}$. Note that $(\bar{R}_k)$ are related to $(\hat{R}_k)$ via simple inequalities $\hat{R}_1 \leq \bar{R}_1 \leq 2\hat{R}_1$ and $\hat{R}_k - \hat{R}_{k-1} - 1 \leq \bar{R}_k \leq 2\hat{R}_k$ for $k \geq 2$. The next theorem immediately follows from Theorem 3.1.6 and the fact that $\mathbb{P}_{cr}(0 \leftrightarrow \partial B(n)) \asymp \mathbb{P}_{cr}(0 \leftrightarrow \partial B(2n))$.

Theorem 3.1.9. *For every $k \geq 1$,*

$$\mathbb{P}(\bar{R}_k \geq n) \asymp (\log n)^{k-1}\mathbb{P}_{cr}(0 \leftrightarrow \partial B(n)).$$

Finally, we give the corresponding result for the volume of the kth pond. Let $\hat{V}_k$ be the number of vertices in G_k.

Theorem 3.1.10. *For any $k \geq 1$,*

$$\mathbb{P}(\hat{V}_k \geq n) \asymp (\log n)^{k-1}\mathbb{P}_{cr}(|\mathcal{C}(0)| \geq n).$$

Mutual singularity of IPC and IIC

First we recall the definition of the incipient infinite cluster from [33]. It is shown in [33] that the limit

$$\nu(E) = \lim_{N\to\infty} \mathbb{P}_{cr}(E \mid 0 \leftrightarrow \partial B(N))$$

exists for any event E that depends on the state of finitely many edges in $\mathbb{E}^2$. The unique extension of ν to a probability measure on configurations of open and closed edges exists. Under this measure, the open cluster of the origin is a.s. infinite. It is called the *incipient infinite cluster* (IIC). Recall the definition of the IPC $\mathcal{S}$ from Section 3.1.1. The next statement is [30, Theorem 3].

Theorem 3.1.11. *For any finite $\mathcal{K} \subset \mathbb{E}^2$ and $x \in \mathbb{Z}^2$, let $\mathcal{K}(x) = x + \mathcal{K} \subset \mathbb{E}^2$, $E_{\mathcal{K}} = \{\mathcal{K} \subset \mathcal{S}\}$ and $E'_{\mathcal{K}} = \{\mathcal{K} \subset \mathcal{C}(0)\}$. Then*

$$\lim_{|x|\to\infty} \mathbb{P}(E_{\mathcal{K}(x)} \mid x \in \mathcal{S}) = \nu(E'_{\mathcal{K}}).$$

The above theorem says that asymptotically the distribution of invaded edges near x is given by the IIC measure. In this chapter we show that globally the IPC measure and the IIC measure are entirely different.

Theorem 3.1.12. *The laws of IPC and IIC are mutually singular.*

3.1.4 Structure of this chapter

We define the correlation length and state some of its properties in Section 3.2. We prove Theorem 3.1.1 in Section 3.3 and Theorem 3.1.3 in Section 3.4. The proof of Theorem 3.1.4 and Theorem 3.1.5 are given in Section 3.5. In Section 3.6 we prove Theorem 3.1.6. Theorem 3.1.12 is proven in Section 3.7. After Sections 3.1 and 3.2, the remainder of the chapter may be read in any order. For the notation in Sections 3.3-3.7 we refer the reader to Section 3.1.2.

3.2 Correlation length and preliminary results

In this section we define the correlation length that will play a crucial role in our proofs. The correlation length was introduced in [20] and further studied in [34].

3.2.1 Correlation length

For m, n positive integers and $p \in (p_c, 1]$ let

$$\sigma(n, m, p) = \mathbb{P}_p(\text{there is an open horizontal crossing of } [0, n] \times [0, m]).$$

Given $\varepsilon > 0$, we define

$$L(p, \varepsilon) = \min\{n : \ \sigma(n, n, p) \geq 1 - \varepsilon\}. \tag{3.2.1}$$

$L(p, \varepsilon)$ is called the finite-size scaling correlation length and it is known that $L(p, \varepsilon)$ scales like the usual correlation length (see [34]). Therefore, we will refer to $L(p, \varepsilon)$ simply as the correlation length. It was also shown in [34] that the scaling of $L(p, \varepsilon)$ is independent of ε given that it is small enough, i.e. there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon_1, \varepsilon_2 \leq \varepsilon_0$ we have $L(p, \varepsilon_1) \asymp L(p, \varepsilon_2)$. For simplicity we will write $L(p) = L(p, \varepsilon_0)$ for the entire chapter. We also define

$$p_n = \sup\{p : \ L(p) > n\}.$$

It is easy to see that $L(p) \to \infty$ as $p \to p_c$ and $L(p) = 1$ for p close to 1. In particular, the probability p_n is well-defined. It is clear from the definitions of $L(p)$ and p_n and from the RSW theorem that, for positive integers k and l, there exists $\delta_{k,l} > 0$ such that, for any positive integer n and for all $p \in [p_c, p_n]$,

$$\mathbb{P}_p(\text{there is an open horizontal crossing of } [0, kn] \times [0, ln)) > \delta_{k,l}$$

and

$$\mathbb{P}_p(\text{there is a closed horizontal dual crossing of } ([0, kn] \times [0, ln))^*) > \delta_{k,l}.$$

By the FKG inequality and a standard gluing argument [26, Section 11.7] we get that, for positive integers n and $k \geq 2$ and for all $p \in [p_c, p_n]$,

$$\mathbb{P}_p(Ann(n, kn) \text{ contains an open circuit around the origin}) > (\delta_{2k,k-1})^4$$

and

$$\mathbb{P}_p(Ann(n, kn)^* \text{ contains a closed dual circuit around the origin}) > (\delta_{2k,k-1})^4.$$

3.2.2 Preliminary results

For any positive l we define $\log^{(0)} l = l$ and $\log^{(j)} l = \log(\log^{(j-1)} l)$ for all $j \geq 1$, as long as the right-hand side is well defined. For $l > 10$, let

$$\log^* l = \min\{j > 0 \ : \ \log^{(j)} l \text{ is well-defined and} \log^{(j)} l \leq 10\}. \tag{3.2.2}$$

Our choice of the constant 10 is quite arbitrary, we could take any other large enough positive number instead of 10. For $l > 10$, let

$$p_l(j) = \begin{cases} \inf\left\{p > p_c \ : \ L(p) \le \frac{l}{C_* \log^{(j)} l}\right\} & \text{if } j \in (0, \log^* l), \\ p_c & \text{if } j \ge \log^* l, \\ 1 & \text{if } j = 0. \end{cases} \tag{3.2.3}$$

The value of C_* will be chosen later. Note that there exists a universal constant $L_0(C_*) > 10$ such that $p_l(j)$ are well-defined if $l > L_0(C_*)$ and non-increasing in l. The last observation follows from monotonicity of $L(p)$ and the fact that the functions $l/\log^{(j)} l$ are non-decreasing in l for $j \in (0, \log^* l)$ and $l \ge 3$.

We give the following results without proofs.

1. There is a constant D such that
$$\lim_{\delta \downarrow 0} \frac{L(p-\delta)}{L(p)} \le D \quad \forall\, p > p_c. \tag{3.2.4}$$
2. ([30, (2.10)]) There exists a universal constant D_1 such that, for every $l > L_0(C_*)$ and $j \in (0, \log^* l)$,
$$C_* \log^{(j)} l \le \frac{l}{L(p_l(j))} \le D_1 C_* \log^{(j)} l. \tag{3.2.5}$$
3. ([34, Theorem 2]) There is a constant D_2 such that, for all $p > p_c$,
$$\theta(p) \le \mathbb{P}_p\big[0 \leftrightarrow \partial B(L(p))\big] \le D_2 \mathbb{P}_{cr}\big[0 \leftrightarrow \partial B(L(p))\big], \tag{3.2.6}$$
where $\theta(p) = \mathbb{P}_p(0 \leftrightarrow \infty)$ is the percolation function for Bernoulli percolation.
4. ([42, Section 4]) There is a constant D_3 such that, for all $n \ge 1$,
$$\mathbb{P}_{p_n}(B(n) \leftrightarrow \infty) \ge D_3. \tag{3.2.7}$$
5. ([34, (3.61)]) There is a constant D_4 such that, for all positive integers $r \le s$,
$$\frac{\mathbb{P}_{cr}(0 \leftrightarrow \partial B(s))}{\mathbb{P}_{cr}(0 \leftrightarrow \partial B(r))} \ge D_4 \sqrt{\frac{r}{s}}. \tag{3.2.8}$$
6. Recall that B_n is the event that there is a closed circuit around the origin in the dual lattice with diameter at least n. There exist positive constants D_5 and D_6 such that, for all $p > p_c$,
$$\mathbb{P}_p(B_n) \le D_5 \exp\left\{-D_6 \frac{n}{L(p)}\right\}. \tag{3.2.9}$$
It follows, for example, from [30, (2.6) and (2.8)] (see also [44, Lemma 39 and Remark 40]).
7. (See for instance [44, Proposition 34]) Fix $e = \langle (0,0), (1,0) \rangle$, and let $A_n^{2,2}$ be the event that e_x and e_y are connected to $\partial B(n)$ by open paths, and e_x^* and e_y^* are connected to $\partial B(n)^*$ by closed paths. Note that these four paths are disjoint and alternate. Then
$$(p_n - p_c) n^2 \mathbb{P}_{cr}(A_n^{2,2}) \asymp 1, \qquad n \ge 1. \tag{3.2.10}$$

3.3 Proof of Theorem 3.1.1

Before we prove Theorem 3.1.1, we give two lemmas that will be used in the proof. To simplify the notations we write $0 = x_0$. For positive integers $m < n$ and $x \in \mathbb{Z}^2$, we define the event

$$A_{m,n}(x) = \{\text{there is an open circuit in the annulus } Ann(x; m, n)\}. \quad (3.3.1)$$

Lemma 3.3.1. *Given a set of vertices* $\{x_1, \ldots, x_k\} \in \mathbb{Z}^2$, *let* $m_i = \min\{|x_i - x_j| : 0 \leq j \leq k, j \neq i\}$, *where* $x_0 = 0$ *and let* $m = \min\{m_i : 0 \leq i \leq k\}$. *Furthermore, assume* $m = m_k$. *Then there exists a constant* C_1, *independent of* k, *such that for all* $p > p_c$ *the probability*

$$\mathbb{P}_p(x_1 \leftrightarrow \infty, \ldots, x_k \leftrightarrow \infty, 0 \leftrightarrow \infty)$$

is bounded from above by

$$C_1 \mathbb{P}_p(x_k \leftrightarrow \partial B(x_k, m)) \mathbb{P}_p(x_1 \leftrightarrow \infty, \ldots, x_{k-1} \leftrightarrow \infty, 0 \leftrightarrow \infty).$$

Proof of Lemma 3.3.1. The statement is trivial if $m \leq 4$, therefore we assume that $m > 4$. By Russo-Seymour-Welsh theorem, there is a constant C_2 independent of k and m such that, for all $p > p_c$, $\mathbb{P}_p(A_{\lfloor m/4 \rfloor, \lfloor m/2 \rfloor}(x_k)) \geq 1/C_2$, and hence $1 \leq C_2 \mathbb{P}_p(A_{\lfloor m/4 \rfloor, \lfloor m/2 \rfloor}(x_k))$. The FKG inequality gives

$$\begin{aligned} &\mathbb{P}_p(x_1 \leftrightarrow \infty, \ldots, x_k \leftrightarrow \infty, 0 \leftrightarrow \infty) \\ &\leq C_2 \mathbb{P}_p(A_{\lfloor m/4 \rfloor, \lfloor m/2 \rfloor}(x_k)) \mathbb{P}_p(x_1 \leftrightarrow \infty, \ldots, x_k \leftrightarrow \infty, 0 \leftrightarrow \infty) \\ &\leq C_2 \mathbb{P}_p(A_{\lfloor m/4 \rfloor, \lfloor m/2 \rfloor}(x_k), x_1 \leftrightarrow \infty, \ldots, x_k \leftrightarrow \infty, 0 \leftrightarrow \infty). \quad (3.3.2) \end{aligned}$$

The event on the right hand side of (3.3.2) implies the following two events

1. $\{x_k \leftrightarrow \partial B(x_k, \lfloor m/4 \rfloor)\}$, and
2. $\{x_1 \leftrightarrow \infty, \ldots, x_{k-1} \leftrightarrow \infty, 0 \leftrightarrow \infty \text{ outside } B(x_k, \lfloor m/4 \rfloor)\}$.

These two events are independent and therefore the right hand side of (3.3.2) is bounded from above by

$$C_2 \mathbb{P}_p(x_k \leftrightarrow \partial B(x_k, \lfloor m/4 \rfloor)) \mathbb{P}_p(x_1 \leftrightarrow \infty, \ldots, x_{k-1} \leftrightarrow \infty, 0 \leftrightarrow \infty \text{ outside } B(x_k, \lfloor m/4 \rfloor))$$

$$\leq C_2 \mathbb{P}_p(x_k \leftrightarrow \partial B(x_k, \lfloor m/4 \rfloor)) \mathbb{P}_p(x_1 \leftrightarrow \infty, \ldots, x_{k-1} \leftrightarrow \infty, 0 \leftrightarrow \infty),$$

where the last inequality follows from monotonicity. Finally, it follows from the FKG inequality, RSW theorem and a standard gluing argument [26, Section 11.7] that $\mathbb{P}_p(x_k \leftrightarrow \partial B(x_k, \lfloor m/4 \rfloor)) \asymp \mathbb{P}_p(x_k \leftrightarrow \partial B(x_k, m))$ uniformly in $p > p_c$. □

We recall the definition of the probabilities $(p_n(j))$ in (3.2.3). We also recall that these probabilities are well defined if $n > L_0(C_*)$, where C_* is the constant from (3.2.3). Later we choose C_* large enough.

Lemma 3.3.2. *Given a set of vertices $\{x_1, \ldots, x_k\} \in \mathbb{Z}^2$, let $n = \max\{|x_i - x_j| : i, j = 0 \ldots k\}$, where $x_0 = 0$. Furthermore, assume that $n \geq L_0(C_*)$. Then there is a universal constant C_3 such that, for all $j \in (0, \log^* n)$,*

$$\mathbb{P}_{p_n(j)}(x_1 \leftrightarrow \infty, \ldots, x_k \leftrightarrow \infty, 0 \leftrightarrow \infty) \leq (C_3 \log^{(j)} n)^{\frac{k+1}{2}} \mathbb{P}_{cr}(x_1, \ldots, x_k \in \mathcal{C}(0)). \tag{3.3.3}$$

Proof of Lemma 3.3.2. We will use induction in k. First, we consider the case $k = 1$. To simplify our notation we write $x_1 = x$. Note that now $|x| = n = m$, where m is defined as in Lemma 3.3.1. From Lemma 3.3.1 it follows that

$$\mathbb{P}_{p_n(j)}(x \leftrightarrow \infty, 0 \leftrightarrow \infty) \leq C_1 \theta(p_n(j)) \mathbb{P}_{p_n(j)}(0 \leftrightarrow \partial B(n)). \tag{3.3.4}$$

Since $L(p_n(j)) \leq n$, we obtain

$$\mathbb{P}_{p_n(j)}(0 \leftrightarrow \partial B(n)) \leq \mathbb{P}_{p_n(j)}(0 \leftrightarrow \partial B(L(p_n(j)))).$$

Combined with (3.2.5), (3.2.6) and (3.2.8), the above inequality gives

$$C_1 \theta(p_n(j)) \mathbb{P}_{p_n(j)}(0 \leftrightarrow \partial B(n)) \leq C_4 \mathbb{P}_{cr}(0 \leftrightarrow \partial B(L(p_n(j))))^2$$

$$\leq C_5 \frac{n}{L(p_n(j))} \mathbb{P}_{cr}(0 \leftrightarrow \partial B(n))^2 \leq C_6 \log^{(j)} n \mathbb{P}_{cr}(0 \leftrightarrow \partial B(n))^2.$$

RSW theorem and the gluing argument show (see e.g. [32, (4)]) that

$$\mathbb{P}_{cr}(0 \leftrightarrow \partial B(n))^2 \leq C_7 \mathbb{P}_{cr}(x \in \mathcal{C}(0)), \tag{3.3.5}$$

with some constant C_7. In particular, (3.3.3) follows for $k = 1$.

The general case is more involved. We assume that Lemma 3.3.2 is proven for any set of vertices $\{y_1, \ldots, y_{k-1}\} \in \mathbb{Z}^2$. Then for a set of vertices $\{x_1, \ldots, x_k\} \in \mathbb{Z}^2$ we define m as in Lemma 3.3.1 and assume that $m = m_k = \min\{|x_i - x_k| : \ i < k\}$. We also define $n_1 = \max\{|x_i - x_j| : i, j = 0 \ldots k-1\}$, with $x_0 = 0$. Then, by the induction hypothesis

$$\mathbb{P}_{p_{n_1}(j)}(x_1 \leftrightarrow \infty, \ldots, x_{k-1} \leftrightarrow \infty, 0 \leftrightarrow \infty) \leq (C_3 \log^{(j)} n_1)^{\frac{k}{2}} \mathbb{P}_{cr}(x_1, \ldots, x_{k-1} \in \mathcal{C}(0)). \tag{3.3.6}$$

Since $n_1 \leq n$ and $m \leq n$, we get $p_n(j) \leq p_m(j)$ and $p_n(j) \leq p_{n_1}(j)$ (see Section 3.2). Therefore,

$$\begin{aligned}
&\mathbb{P}_{p_n(j)}(x_1 \leftrightarrow \infty, \ldots, x_k \leftrightarrow \infty, 0 \leftrightarrow \infty) \\
&\leq C_1 \mathbb{P}_{p_m(j)}(x_k \leftrightarrow \partial B(x_k, m)) \mathbb{P}_{p_{n_1}(j)}(x_1 \leftrightarrow \infty, \ldots, x_{k-1} \leftrightarrow \infty, 0 \leftrightarrow \infty) \\
&\leq C_1 \mathbb{P}_{p_m(j)}(x_k \leftrightarrow \partial B(x_k, m)) (C_3 \log^{(j)} n_1)^{\frac{k}{2}} \mathbb{P}_{cr}(x_1, \ldots, x_{k-1} \in \mathcal{C}(0)) \\
&\leq (C_8 \log^{(j)} m)^{1/2} \mathbb{P}_{cr}(x_k \leftrightarrow \partial B(x_k, m)) (C_3 \log^{(j)} n_1)^{\frac{k}{2}} \mathbb{P}_{cr}(x_1, \ldots, x_{k-1} \in \mathcal{C}(0)) \\
&\leq C_8^{\frac{1}{2}} C_3^{\frac{k}{2}} (\log^{(j)} n)^{\frac{k+1}{2}} \mathbb{P}_{cr}(x_k \leftrightarrow \partial B(x_k, m)) \mathbb{P}_{cr}(x_1, \ldots, x_{k-1} \in \mathcal{C}(0)),
\end{aligned}$$

where the first inequality follows from Lemma 3.3.1 and monotonicity, the second inequality follows from (3.3.6), the third inequality follows from (3.2.5) and (3.2.8). Note that C_8 is independent of k. Now it suffices to show that there is a universal constant C_9 such that

$$\mathbb{P}_{cr}(x_k \leftrightarrow \partial B(x_k, m))\mathbb{P}_{cr}(x_1, \dots, x_{k-1} \in \mathcal{C}(0)) \leq C_9 \mathbb{P}_{cr}(x_1, \dots, x_k \in \mathcal{C}(0)). \tag{3.3.7}$$

Assume that (3.3.7) is proven. Then we can take $C_3 = \max\{C_6C_7, C_8C_9^2\}$. The argument above shows that we can proceed to the next k using this value of C_3. Now we show (3.3.7). We take x_i such that $m = |x_k - x_i|$. Note that this vertex may be the origin. We know that at least one such vertex exists. Recall the definition of events $A_{m,n}(x)$ from (3.3.1). By the RSW theorem, there is a constant C_{10} such that $1 \leq C_{10}\mathbb{P}_{cr}(A_{\lfloor m/2 \rfloor, m}(x_i); A_{\lfloor m/2 \rfloor, m}(x_k))$. Using the FKG inequality we get

$$\begin{aligned}
&\mathbb{P}_{cr}(x_k \leftrightarrow \partial B(x_k, m))\mathbb{P}_{cr}(x_1, \dots, x_{k-1} \in \mathcal{C}(0)) \\
&\leq C_{10}\mathbb{P}_{cr}(A_{\lfloor m/2 \rfloor, m}(x_i); A_{\lfloor m/2 \rfloor, m}(x_k))\mathbb{P}_{cr}(x_k \leftrightarrow \partial B(x_k, m))\mathbb{P}_{cr}(x_1, \dots, x_{k-1} \in \mathcal{C}(0)) \\
&\leq C_{10}\mathbb{P}_{cr}(A_{\lfloor m/2 \rfloor, m}(x_i); A_{\lfloor m/2 \rfloor, m}(x_k); x_k \leftrightarrow \partial B(x_k, m); x_1, \dots, x_{k-1} \in \mathcal{C}(0)).
\end{aligned}$$

We show that the event

$$\{A_{\lfloor m/2 \rfloor, m}(x_i); A_{\lfloor m/2 \rfloor, m}(x_k); x_k \leftrightarrow \partial B(x_k, m); x_1, \dots, x_{k-1} \in \mathcal{C}(0)\}$$

implies the event $\{x_i \leftrightarrow x_k; x_1, \dots, x_{k-1} \in \mathcal{C}(0)\}$. Indeed, it follows from simple observations:

1. Since the events $\{x_k \leftrightarrow \partial B(x_k, m)\}$ and $A_{\lfloor m/2 \rfloor, m}(x_k)$ hold, x_k is connected to the circuit lying in the annulus $Ann(x_k; \lfloor m/2 \rfloor, m)$.
2. Since the distance between x_i and x_k is m, the boxes $B(x_i, \lfloor m/2 \rfloor + 1)$ and $B(x_k, \lfloor m/2 \rfloor + 1)$ intersect. This implies that the circuits in the annuli $Ann(x_k; \lfloor m/2 \rfloor, m)$ and $Ann(x_i; \lfloor m/2 \rfloor, m)$ intersect.
3. Recall that m is the minimal distance in the graph with vertex set $\{0, x_1, \dots, x_k\}$. Since $k \geq 2$ and $\{x_1, \dots, x_{k-1} \in \mathcal{C}(0)\}$, there is a vertex $x_j \neq x_k$ (it may be the origin) such that $x_j \notin B(x_i, m-1)$ and x_j is connected to x_i. The last observation implies that x_i is connected to the circuit lying in $Ann(x_i; \lfloor m/2 \rfloor, m)$ and hence also to x_k.

This proves (3.3.7). □

Proof of Theorem 3.1.1. For $\{x_1, \dots, x_k\} \in \mathbb{Z}^2$ we define, as in Lemma 3.3.2, $n = \max\{|x_i - x_j| : i, j = 0 \dots k\}$. If $n < L_0(C_*)$ then $\mathbb{P}_{cr}(x_1, \dots, x_k \in \mathcal{C}(0)) > \text{const}(C_*)$. Theorem 3.1.1 immediately follows, since $\mathbb{P}(x_1, \dots, x_k \in \hat{V}_1) \leq 1$. Therefore we can assume that $n \geq L_0(C_*)$. In particular, the probabilities $p_n(j)$ are well-defined. Recall that $\hat{\tau}_1$ is the value of the outlet of the first pond. We decompose the event $\{x_1, \dots x_k \in \hat{V}_1\}$ according to the value of $\hat{\tau}_1$. We write

$$\mathbb{P}(x_1, \dots, x_k \in \hat{V}_1) = \sum_{j=1}^{\log^* n} \mathbb{P}\left(x_1, \dots, x_k \in \hat{V}_1, \hat{\tau}_1 \in [p_n(j), p_n(j-1))\right). \tag{3.3.8}$$

Note that, for any $p > p_c$,

(a) if $\hat{\tau}_1 < p$, then any invaded site is in the infinite p-open cluster;
(b) if a given set of vertices $\{x_1 \dots x_k\}$ is in the first pond, n is defined as in Lemma 3.3.2 and $\hat{\tau}_1 > p$, then there is a p-closed circuit around the origin with diameter at least n.

We recall the definition of the event

$$B_{n,p} = \{\exists\ p\text{-closed circuit around } 0 \text{ in the dual with diameter at least } n\}.$$

We conclude that the probability $\mathbb{P}(x_1, \dots, x_k \in \hat{V}_1, \hat{\tau}_1 \in [p_n(j), p_n(j-1)))$ is bounded from above by

$$\mathbb{P}\left(x_1 \overset{p_n(j-1)}{\longleftrightarrow} \infty, \dots, x_k \overset{p_n(j-1)}{\longleftrightarrow} \infty, 0 \overset{p_n(j-1)}{\longleftrightarrow} \infty; B_{n,p_n(j)}\right). \tag{3.3.9}$$

The FKG inequality implies that the probability (3.3.9) is not bigger than

$$\begin{aligned}&\mathbb{P}_{p_n(j-1)}(x_1 \leftrightarrow \infty, \dots, x_k \leftrightarrow \infty, 0 \leftrightarrow \infty)\,\mathbb{P}(B_{n,p_n(j)})\\ &\le C_{11}(\log^{(j-1)} n)^{-C_{12}}\mathbb{P}_{p_n(j-1)}(x_1 \leftrightarrow \infty, \dots, x_k \leftrightarrow \infty, 0 \leftrightarrow \infty)\end{aligned} \tag{3.3.10}$$

where we use (3.2.5) and (3.2.9) to bound the probability of $B_{n,p_n(j)}$ by $C_{11}(\log^{(j-1)} n)^{-C_{12}}$. The constant C_{12} can be made arbitrarily large given that C_* is made large enough. We consider bounds for (3.3.10) separately for $j = 1$ and for $j > 1$. If $j > 1$, we use Lemma 3.3.2 to bound (3.3.10) by

$$C_{13}(\log^{(j-1)} n)^{\frac{k+1}{2} - C_{12}}\mathbb{P}_{cr}(x_1, \dots, x_k \in \mathcal{C}(0)).$$

If $j = 1$, we bound (3.3.10) by

$$C_{11}n^{-C_{12}} \le C_{14}n^{-\frac{1}{2}}\mathbb{P}_{cr}(0 \leftrightarrow \partial B(n))^{2k} \le C_{15}n^{-\frac{1}{2}}\mathbb{P}_{cr}(x_1, \dots, x_k \in \mathcal{C}(0)).$$

The first inequality holds for $C_{12} \ge k + 1/2$, since $\mathbb{P}_{cr}(0 \leftrightarrow \partial B(n)) > \frac{1}{2}n^{-\frac{1}{2}}$ (see [26, (11.90)]). The last inequality follows from (3.3.5), applied k times, and the FKG inequality. Therefore, for all j, if $C_{12} \ge k + 1/2$ then (3.3.10) is bounded by

$$C_{16}(\log^{(j-1)} n)^{-\frac{1}{2}}\mathbb{P}_{cr}(x_1, \dots, x_k \in \mathcal{C}(0)).$$

We plug this bound into (3.3.8)

$$\begin{aligned}\mathbb{P}(x_1, \dots, x_k \in \hat{V}_1) &\le C_{16}\mathbb{P}_{cr}(x_1, \dots, x_k \in \mathcal{C}(0)) \sum_{j=1}^{\log^* n} (\log^{(j-1)} n)^{-\frac{1}{2}}\\ &\le C_{17}\mathbb{P}_{cr}(x_1, \dots, x_k \in \mathcal{C}(0)).\end{aligned}$$

The last inequality follows from the fact that

$$\sup_{n>10} \sum_{j=1}^{\log^* n} \left(\log^{(j-1)} n\right)^{-\frac{1}{2}} < \infty \tag{3.3.11}$$

To show (3.3.11) we follow [30] (see [30, (2.26)]). Recall from the definitions that $\log^{(j)} n > 2$. Applying this to the case $j = \log^* n$ shows that the last term in the sum in (3.6.10) is at most $(e^2)^{-1/2}$. Similarly, the penultimate term is at most $(\exp(e^2))^{-1/2}$, etc. This leads to the finite upper bound $\frac{1}{\sqrt{e^2}} + \frac{1}{\sqrt{e^{e^2}}} + \dots$ for the l.h.s. of (3.3.11). $\square$

3.4 Proof of Theorem 3.1.3

In this section we prove that

$$\lim_{n\to\infty} \frac{\mathbb{P}(B(n) \subset \hat{V}_1)}{\mathbb{P}_{cr}(B(n) \subset \mathcal{C}(0))} = \infty.$$

By RSW arguments [26, Section 11.7], the denominator is at most equal to

$$C_1 \mathbb{P}_{cr}(B(n) \subset \mathcal{C}(0) \text{ in } B(2n))$$

for some $C_1 > 0$. Recall that $p_n = \sup\{p : L(p) > n\}$. We can bound the numerator from below: it is at least equal to

$$\mathbb{P}_{p_n}(B(n) \subset \mathcal{C}(0) \text{ in } B(2n) \cap \exists \text{ closed circuit around } B(2n))$$

$$= \mathbb{P}_{p_n}(B(n) \subset \mathcal{C}(0) \text{ in } B(2n)) \mathbb{P}_{p_n}(\exists \text{ closed circuit around } B(2n))$$

By the definition of $L(p)$, there exists $C_2 > 0$ so that this probability is at least

$$C_2 \mathbb{P}_{p_n}(B(n) \subset \mathcal{C}(0) \text{ in } B(2n))$$

Therefore to prove Theorem 3.1.3 it suffices to show

$$\lim_{n\to\infty} \frac{\mathbb{P}_{p_n}(B(n) \subset \mathcal{C}(0) \text{ in } B(2n))}{\mathbb{P}_{cr}(B(n) \subset \mathcal{C}(0) \text{ in } B(2n))} = \infty. \tag{3.4.1}$$

For this we use Russo's formula [26] (the definition of pivotal edges is also given in [26]). Let Γ_n be the event which appears both in the numerator and in the denominator of (3.4.1). Let $p \in [\epsilon, 1-\epsilon]$ for some $\epsilon < \frac{1}{2}$, and, for any vertex v, let E_v be the set of edges incident to v. We see that

$$\begin{aligned}
\frac{d}{dp}\mathbb{P}_p(\Gamma_n) &= \sum_e \mathbb{P}_p(e \text{ is pivotal for } \Gamma_n) \\
&\geq \frac{1}{2p} \sum_{v \in B(n)} \sum_{e \in E_v} \mathbb{P}_p(e \text{ is pivotal for } \Gamma_n; \Gamma_n) \\
&\geq \frac{1}{2p} \sum_{v \in B(n)} \mathbb{P}_p(\exists e \in E_v \text{ pivotal for } \Gamma_n; \Gamma_n) \\
&\geq \frac{1}{2p} \sum_{v \in B(n)} \min(p, 1-p)^4 \mathbb{P}_p(\Gamma_n) \\
&\geq C_3 n^2 \mathbb{P}_p(\Gamma_n).
\end{aligned}$$

In particular,

$$\mathbb{P}_{p_n}(\Gamma_n) \geq \mathbb{P}_{cr}(\Gamma_n) e^{C_4 n^2 (p_n - p_c)}$$

for some $C_4 > 0$. It follows from (3.2.10) and the fact that $\theta(p_c) = 0$ that $n^2(p_n - p_c) \to \infty$. This completes the proof. $\square$

3.5 Proof of Theorem 3.1.4 and Theorem 3.1.5

First we prove two lemmas (see Section 3.1.2 for the definitions).

Lemma 3.5.1. *For each $k \geq 2$ there exists c_k so that, for all n,*

$$\mathbb{P}(A^1_{n,kn,p_c}) \leq c_k,$$

where $c_k \to 0$ as $k \to \infty$.

Proof. Recall that $B_{n,2n} = \{\text{there is a closed circuit in } Ann(n,2n)^*\}$. Pick $c > 0$ so that, for all $N \geq 1$,

$$\mathbb{P}_{cr}(B_{N,2N}) \geq c.$$

We split the annulus $Ann(n,kn)$ into $\lfloor \log k \rfloor$ disjoint annuli $Ann(2^i n, 2^{i+1} n)$:

$$\mathbb{P}(A^1_{n,kn,p_c}) \leq (1-c)^{\log k - 1}.$$

This completes the proof. □

Lemma 3.5.2. *There exists $C_1 > 0$ such that, for all N and k,*

$$\mathbb{P}(A_{N,2N,p_c} \cap A_{kN,2kN,p_c} \cap A^1_{N,2kN,p_N}) \geq C_1.$$

Proof. By RSW arguments, there exists $C_2 > 0$ so that, for all N and k,

$$\mathbb{P}(A_{N,2N,p_c} \cap A_{kN,2kN,p_c}) \geq C_2.$$

If follows from (3.2.7) that there exists $C_3 > 0$ so that, for all N and k,

$$\mathbb{P}(A^1_{N,2kN,p_N}) \geq \mathbb{P}_{p_N}(B(N) \leftrightarrow \infty) \geq C_3.$$

The FKG inequality gives the result. □

Now we prove the theorems.

Proof of Theorem 3.1.4. We prove the theorem for $K = 2$. For other values of K the proof is similar. Let $D(k,N) = A_{N,2N,p_c} \cap A_{kN,2kN,p_c} \cap A^1_{N,2kN,p_N}$ and pick C_1 from Lemma 3.5.2. Fix k so that the constant $c_{k/2}$ from Lemma 3.5.1 satisfies $c_{k/2} \leq \frac{C_1}{2}$. It follows that

$$\mathbb{P}(D(k,N) \cap \{A^1_{2N,kN,p_c}\}^c) \geq \frac{C_1}{2}.$$

For any $k \geq 2$, there exists $C_4 = C_4(k)$ so that, for all N,

$$\mathbb{P}(B_{2kN,4kN,p_N}) \geq C_4.$$

Therefore, by independence,

$$\mathbb{P}(D(k,N) \cap \{A^1_{2N,kN,p_c}\}^c \cap B_{2kN,4kN,p_N}) \geq \frac{C_1 C_4}{2} > 0.$$

This statement, along with Borel-Cantelli's lemma, gives the theorem. □

Proof of Theorem 3.1.5. Let $A^1_{n,p} = \{0 \leftrightarrow \partial B(n) \text{ by a } p-\text{open path}\}$. We first notice that Theorem 3.1.6 gives a constant $C_5 > 0$ so that, for all N,

$$\mathbb{P}(\hat{R}_1 \geq N) \leq C_5 \mathbb{P}(A^1_{2N,p_c}).$$

It is obvious that $\mathbb{P}(\hat{R}_1 \geq N \cap \mathcal{U}(1,2,N)) \geq \mathbb{P}(A^1_{2N,p_c} \cap \mathcal{U}(1,2,N))$. Therefore it suffices to show that there is an $\varepsilon > 0$ so that for all N,

$$\mathbb{P}(\mathcal{U}(1,2,N) | A^1_{2N,p_c}) \geq \varepsilon.$$

The rest of the proof is almost the same as the proof of Theorem 3.1.4. Let $D(k,N)$ be as in the proof of Theorem 3.1.4. Pick C_1 from Lemma 3.5.2. By the FKG inequality, we see that

$$\mathbb{P}(D(k,N) \cap A^1_{2N,p_c}) \geq C_1 \mathbb{P}(A^1_{2N,p_c}).$$

By independence and Lemma 3.5.1, we may fix k so that for all N,

$$\mathbb{P}(A^1_{2N,p_c} \cap A^1_{2N,kN,p_c}) \leq c_{k/2} \mathbb{P}(A^1_{2N,p_c}) \leq \frac{C_1}{2} \mathbb{P}(A^1_{2N,p_c}).$$

For any $k \geq 2$, there exists $C_4 = C_4(k)$ so that, for all N,

$$\mathbb{P}(B_{2kN,4kN,p_N}) \geq C_4.$$

Now independence gives us

$$\mathbb{P}(A^1_{2N,p_c} \cap D(k,N) \cap \{A^1_{2N,kN,p_c}\}^c \cap B_{2kN,4kN,p_N}) \geq \frac{C_1 C_4}{2} \mathbb{P}(A^1_{2N,p_c}).$$

This concludes the proof.

□

3.6 Proof of Theorem 3.1.6 and Theorem 3.1.10

3.6.1 Upper bound in Theorem 3.1.6

We give the proof for $k = 1$ and $k = 2$. The $k = 1$ case shows the basic ideas of the proof for the general k while the $k = 2$ case explains how this ideas can be extended to consider other ponds. The proof for $k \geq 3$ is similar to the proof for $k = 2$.

Starting with the first pond, note that it suffices to prove this for the case that n is of the form 2^k. Indeed, if it holds for those special cases then, for any $2^{k-1} < n < 2^k$ we have

$$\begin{aligned} P(\hat{R}_1 \geq n) &\leq P(\hat{R}_1 \geq 2^{k-1}) \leq c P_{cr}\big[0 \longleftrightarrow \partial B(2^{k-1})\big] \\ &\overset{(3.2.8)}{\leq} \bar{c} P_{cr}\big[0 \longleftrightarrow \partial B(2^k)\big] \leq \bar{c} P_{cr}\big[0 \longleftrightarrow \partial B(n)\big]. \end{aligned}$$

Recall the definition of $\log^* k$ and $p_k(j)$ from Section 3.2,

$$p_l(j) = \begin{cases} \inf\Big\{p > p_c \ : \ L(p) \le \frac{l}{C_2 \log^{(j)} l}\Big\} & \text{if } j \in (0, \log^* l), \\ p_c & \text{if } j \ge \log^* l, \\ 1 & \text{if } j = 0. \end{cases} \tag{3.6.1}$$

where the constant C_2 will be chosen later.

Now we decompose $P(\hat{R}_1 \ge n)$ according to the τ–value of the first outlet, $\hat{\tau}_1$, as follows, where we note that since τ has a continuous distribution, $\hat{\tau}_1$ does not coincide with $p_k(j)$ for any $j = 0, \dots, \log^* k$, almost surely.

$$\begin{aligned} P(\hat{R}_1 \ge n) = P(\hat{R}_1 \ge n, p_k(0) < \hat{\tau}_1) + P(\hat{R}_1 \ge n, \hat{\tau}_1 < p_k(\log^* k)) \\ + \sum_{j=0}^{\log^* k - 1} P(\hat{R}_1 \ge n, p_k(j+1) < \hat{\tau}_1 < p_k(j)). \end{aligned} \tag{3.6.2}$$

To bound the terms in (3.6.2) we will use the following observations made in [49]. Let p be an arbitrary number between p_c and 1.

Observations
(a) $\hat{\tau}_1 < p$ if and only if the origin belongs to an infinite p–open cluster.
(b) If $\hat{\tau}_1 > p$ and $\hat{R}_1 \ge n$, then there is a p–closed circuit around O in the dual lattice with diameter at least n.

The event in observation (b) will be denoted by $A_{n,p}$.

$A_{n,p} := \big\{\exists$ p-closed circuit around O in the dual with diameter at least $n\big\}$.

Starting with the first term of (3.6.2), Observation (b) gives

$$P(\hat{R}_1 \ge n, p_k(0) < \hat{\tau}_1) \le P(A_{n,p_k(0)}). \tag{3.6.3}$$

From (3.2.9) in Section 3.2.2 we know that there exist C_3 and C_4 such that for all $p > p_c$,

$$P(A_{n,p}) \le C_3 \exp\Big\{ - \frac{C_4 n}{L(p)} \Big\} \tag{3.6.4}$$

Using the lower bound in (3.2.5) and the definition of $\log^{(0)} k$ we get that

$$P(A_{n,p_k(0)}) \le C_3 \exp\Big\{ - \frac{C_4 n}{L(p_k(0))} \Big\} \overset{(3.2.5)}{\le} C_3 n^{-C_4 C_2} \tag{3.6.5}$$

As mentioned above, we have $P_{cr}(0 \leftrightarrow \partial B(n)) \ge C n^{-1/2}$. Hence, by taking $C_2 \ge 1/C_4$, we can ensure that

$$P(A_{n,p_k(0)}) \le C_3 n^{-1} \le \tilde{C}_3 P_{cr}(0 \leftrightarrow \partial B(n)).$$

Remark: *For future purposes we will even take $C_2 \ge 2/C_4$.*

For the second term of (3.6.2) we apply observation (a) to get

$$P(\hat{R}_1 \geq n, \hat{\tau}_1 < p_k(\log^* k)) \leq P(\hat{\tau}_1 < p_k(\log^* k)) \overset{\text{Obs. (a)}}{\leq} \theta(p_k(\log^* k)).$$

Furthermore, using (3.2.6), (3.2.5), the definition of $p_k(\log^* k)$ and (3.2.8), we have

$$\begin{aligned}\theta(p_k(\log^* k)) &\leq C_1 P_{cr}\big[0 \leftrightarrow \partial B(L(p_k(\log^* k))\big]\\ &\leq C_1 P_{cr}\big[0 \leftrightarrow \partial B(\frac{2^k}{10DC_2})\big]\\ &\leq C_5 P_{cr}\big[0 \leftrightarrow \partial B(n)\big],\end{aligned}$$

for some constant C_5.

Now let us consider a typical term in the summation in (3.6.2). The two observations a few lines below (3.6.2) (and the definition of $A_{n,p}$) give

$$\begin{aligned}&P(\hat{R}_1 \geq n, p_k(j+1) < \hat{\tau}_1 < p_k(j))\\ &\leq P(0 \overset{p_k(j)}{\longleftrightarrow} \infty,\ A_{n,p_k(j+1)})\\ &\leq \theta(p_k(j))P(A_{n,p_k(j+1)}),\end{aligned} \tag{3.6.6}$$

where in the last inequality we use the Harris-FKG inequality [26, Section 2.2]. To bound the first factor in the right hand side of (3.6.6), note that

$$\begin{aligned}\theta(p_k(j)) &\overset{(3.2.6)}{\leq} C_1 P_{cr}(0 \leftrightarrow L(p_k(j)))\\ &= C_1 P_{cr}(0 \leftrightarrow \partial B(2^k))\frac{P_{cr}(0 \leftrightarrow L(p_k(j)))}{P_{cr}(0 \leftrightarrow \partial B(2^k))}\\ &\overset{(3.2.8)}{\leq} \frac{C_1}{D_1} P_{cr}(0 \leftrightarrow \partial B(2^k))\Big(\frac{2^k}{L(p_k(j))}\Big)^{1/2}\\ &\overset{(3.2.5)}{\leq} \frac{C_1}{D_1} P_{cr}(0 \leftrightarrow \partial B(2^k))(DC_2 \log^{(j)} k)^{1/2}.\end{aligned} \tag{3.6.7}$$

The second factor in the right hand side of (3.6.6) can be bounded using (3.6.4), (3.2.5), (3.6.1) and the choice of C_2:

$$P(A_{n,p_k(j+1)}) \leq C_3 \exp\Big\{-\frac{C_4 n}{L(p_k(j+1))}\Big\} \leq C_3(\log^{(j)} k)^{-1}, \tag{3.6.8}$$

Combining (3.6.7) and (3.6.8) gives

$$\theta(p_k(j))P(A_{n,p_k(j+1)}) \leq C_8(\log^{(j)} k)^{-1/2} P_{cr}\big[0 \leftrightarrow \partial B(n)\big]. \tag{3.6.9}$$

To conclude the proof, just as in the proof of Theorem 3.1.1, we note that

$$\sup_{k>10} \sum_{j=0}^{\log^* k-1} (\log^{(j)} k)^{-1/2} = C_9 < \infty. \tag{3.6.10}$$

Putting everything together we get

$$P(\hat{R}_1 \geq n) \leq \big(\tilde{C}_3 + C_5 + C_8 C_9\big) P_{cr}\big[0 \leftrightarrow \partial B(n)\big].$$

The proof for the radius of the second pond is more involved but it is based on similar ideas as the case of the first pond. To simplify our formulas we restart the counting of the constants now. We fix n and divide the box $B(n)$ into $\lfloor \log n \rfloor + 1$ annuli. We write

$$\mathbb{P}(\hat{R}_2 \geq n) = \mathbb{P}(\hat{R}_1 \geq n) + \sum_{k=1}^{\lfloor \log n \rfloor + 1} \mathbb{P}\Big(\hat{R}_2 \geq n, \hat{R}_1 \in \Big[\frac{n}{2^k}, \frac{n}{2^{k-1}}\Big)\Big). \quad (3.6.11)$$

We already have that

$$\mathbb{P}(\hat{R}_1 \geq n) \leq C_1 \mathbb{P}_{cr}(0 \leftrightarrow \partial B(n)) \leq C_1 \log n \mathbb{P}_{cr}(0 \leftrightarrow \partial B(n)).$$

Therefore, it remains to bound the typical term of the sum on the r.h.s. of (3.6.11). It is sufficient to show that there exists a constant C_2 such that, for any $m \in [0, n/2]$,

$$\mathbb{P}(\hat{R}_2 \geq n; \hat{R}_1 \in [m, 2m]) \leq C_2 \mathbb{P}_{cr}(0 \leftrightarrow \partial B(n)). \quad (3.6.12)$$

We only consider the case $m \geq L_0(C_*)$. The proof for $m < L_0(C_*)$ is similar to the proof for $m \geq L_0(C_*)$ but much simpler. We omit the details. We now assume that $m \geq L_0(C_*)$. In particular, the probabilities $(p_m(i))$ and $(p_n(j))$ are well-defined.

We decompose the event on the l.h.s. according to the τ value of the first and the second outlet. The probability $\mathbb{P}\big(\hat{R}_2 \geq n; \hat{R}_1 \in [m, 2m]\big)$ is bounded from above by

$$\sum_{i=1}^{\log^* m} \sum_{j=1}^{\log^* n} \mathbb{P}\big(\hat{R}_2 \geq n; \hat{R}_1 \in [m, 2m]; \hat{\tau}_1 \in [p_m(i), p_m(i-1)]; \hat{\tau}_2 \in [p_n(j), p_n(j-1)]\big). \quad (3.6.13)$$

Note that if the event $\{\hat{R}_1 \geq m; \hat{\tau}_1 \in [p_m(i), p_m(i-1)]\}$ occurs then

- there is a $p_m(i-1)$-open path from the origin to infinity, and
- the origin is surrounded by a $p_m(i)$-closed circuit of diameter at least m in the dual lattice.

We also note that if the event $\{\hat{R}_1 \leq 2m; \hat{R}_2 \geq n; \hat{\tau}_2 \in [p_n(j), p_n(j-1)]\}$ occurs then

- there is a $p_n(j-1)$-open path from the box $B(2m)$ to infinity, and
- the origin is surrounded by a $p_n(j)$-closed circuit of diameter at least n in the dual lattice.

From the two observations above, the sum (3.6.13) is less than

$$\sum_{i=1}^{\log^* m} \sum_{j=1}^{\log^* n} \mathbb{P}\big(0 \overset{p_m(i-1)}{\longleftrightarrow} \partial B(m); B(2m) \overset{p_n(j-1)}{\longleftrightarrow} \partial B(n); B_{m,p_m(i)}; B_{n,p_n(j)}\big). \quad (3.6.14)$$

The FKG inequality and the independence of the first two events imply that (3.6.14) is not larger than

$$\sum_{i=1}^{\log^* m}\sum_{j=1}^{\log^* n}\mathbb{P}_{p_m(i-1)}\big(0\leftrightarrow\partial B(m)\big)\mathbb{P}_{p_n(j-1)}\big(B(2m)\leftrightarrow\partial B(n)\big)\mathbb{P}\big(B_{m,p_m(i)};B_{n,p_n(j)}\big). \tag{3.6.15}$$

We use (3.2.5) and (3.2.9) to bound the probability of $B_{m,p_m(i)}$ by $C_3(\log^{(i-1)} m)^{-C_4}$, where C_4 can be made arbitrarily large given that C_* is made large enough. Substitution gives a bound for the last term of (3.6.15):

$$\begin{aligned}\mathbb{P}\big(B_{m,p_m(i)};B_{n,p_n(j)}\big) &\le \min\left[C_3(\log^{(i-1)} m)^{-C_4},C_3(\log^{(j-1)} n)^{-C_4}\right]\\ &= C_3\max\left[\log^{(i-1)} m,\log^{(j-1)} n\right]^{-C_4}\\ &\le C_3\left(\log^{(i-1)} m\right)^{-\frac{C_4}{2}}\left(\log^{(j-1)} n\right)^{-\frac{C_4}{2}}. \end{aligned}\tag{3.6.16}$$

Russo-Seymour-Welsh theorem (see Appendix A) and the FKG inequality imply that

$$\mathbb{P}_p\big(0\leftrightarrow\partial B(m)\big)\mathbb{P}_p\big(B(2m)\leftrightarrow\partial B(n)\big)\le C_5\mathbb{P}_p\big(0\leftrightarrow\partial B(n)\big) \tag{3.6.17}$$

uniformly in $p\ge p_c$. Furthermore, using (3.2.5)–(3.2.8) we get

$$\mathbb{P}_{p_m(i-1)}\big(0\leftrightarrow\partial B(m)\big)\le C_6\big(\log^{(i-1)} m\big)^{\frac{1}{2}}\mathbb{P}_{cr}(0\leftrightarrow\partial B(m)) \tag{3.6.18}$$

and

$$\mathbb{P}_{p_n(j-1)}\big(B(2m)\leftrightarrow\partial B(n)\big)\le C_7\big(\log^{(j-1)} n\big)^{\frac{1}{2}}\mathbb{P}_{cr}\big(B(2m)\leftrightarrow\partial B(n)\big). \tag{3.6.19}$$

In the last inequality we also use (3.6.17). We apply the inequalities (3.6.16), (3.6.17), (3.6.18) and (3.6.19) to (3.6.15). We obtain that the probability $\mathbb{P}(\hat{R}_2\ge n;\hat{R}_1\in[m,2m])$ is not larger than

$$C_8\mathbb{P}_{cr}\big(0\leftrightarrow\partial B(n)\big)\sum_{i=1}^{\log^* m}\sum_{j=1}^{\log^* n}\left(\log^{(i-1)} m\right)^{-\frac{C_4-1}{2}}\left(\log^{(j-1)} n\right)^{-\frac{C_4-1}{2}}.$$

We take C_* large enough so that C_4 is greater than 1. As in (2.26) of [30] it is easy to see that there exists a universal constant $C_9<\infty$ such that, for all $n>10$,

$$\sum_{j=1}^{\log^* n}\left(\log^{(j-1)} n\right)^{-\frac{C_4-1}{2}}\le C_9.$$

□

3.6.2 Lower bound in Theorem 3.1.6

As it is pointed out in [49], it is very easy to see that

$$P(\hat{R}_1 \geq n) \geq P_{cr}(0 \leftrightarrow \partial B(n)), \tag{3.6.20}$$

since the whole p_c-open cluster of the origin is invaded before *any* edge with τ value larger than p_c is added to the invasion cluster. So we can proceed to the second pond.

We first give the main idea of the proof. Recall from Remark 3.1.7 that it is equivalent to prove that $\mathbb{P}(\hat{R}_k \geq n) \geq c_k \mathbb{P}_{cr}(0 \leftrightarrow_{k-1} \partial B(n))$, for some positive constants c_k that do not depend on n. In case $k = 1$ the event $\{0 \xleftrightarrow{p_c} \partial B(n)\}$ obviously implies the event $\{\hat{R}_1 \geq n\}$. However, for $k \geq 2$, the event $\{0 \xleftrightarrow{p_c}_{k-1} \partial B(n)\}$ does not, in general, imply the event $\{\hat{R}_k \geq n\}$. The weights of some defected edges from the definition of the event $\{0 \xleftrightarrow{p_c}_{k-1} \partial B(n)\}$ can be large enough so that these edges are never invaded. We resolve this problem by constructing a subevent of the event $\{0 \xleftrightarrow{p_c}_{k-1} \partial B(n)\}$ which implies the event $\{\hat{R}_k \geq n\}$, and, moreover, the probability of this new event is comparable with the probability $\mathbb{P}(0 \xleftrightarrow{p_c}_{k-1} \partial B(n))$. To construct such an event we first extend results from [34] in Lemma 3.6.5 and Lemma 3.6.7 below. Then we construct events that will be used in the proof of the lower bound in Theorem 3.1.6 and show that they satisfy the desired properties (see, e.g., Corollary 3.6.10 below).

We begin with some definitions and lemmas.

Lemma 3.6.1. *(Generalized FKG) Let $\xi_1, \ldots, \xi_n$ be i.i.d. real valued random variables. Let I_1, I_2, I_3 be disjoint subsets of $\{1, \ldots, n\}$. Let $A_1 \in \sigma(\xi_i : i \in I_1 \cup I_2)$ and $A_2 \in \sigma(\xi_i : i \in I_2)$ be increasing in (ξ_i). Let $B_1 \in \sigma(\xi_i : i \in I_1 \cup I_3)$ and $B_2 \in \sigma(\xi_i : i \in I_3)$ be decreasing in (ξ_i). Then*

$$\mathbb{P}(A_2 \cap B_2 \mid A_1 \cap B_1) \geq \mathbb{P}(A_2)\mathbb{P}(B_2). \tag{3.6.21}$$

Proof. Inequality (3.6.21) for $\mathbb{P}_p$ (rather than $\mathbb{P}$) is given in [34, Lemma 3] or [44, Lemma 13]. The main ingredient of that proof is the Harris-FKG inequality for $\mathbb{P}_p$ (see [26, Theorem (2.4)]), which is also valid for $\mathbb{P}$ (see, e.g., [37, Theorem (5.13)]). Apart from that, the proof of (3.6.21) is analogous to the proof of [34, Lemma 3] and [44, Lemma 13], and we omit it. $\square$

Though we will not apply Lemma 3.6.1 to the following events, they serve as simple examples. The events $\{0 \xleftrightarrow{p} \partial B(n)\}$ and $\{B(m) \xleftrightarrow{p} \partial B(n)\}$ are decreasing in (τ_e), and the events $\{0^* \xleftrightarrow{p^*} \partial B(n)^*\}$, $\{B(m)^* \xleftrightarrow{p^*} \partial B(n)^*\}$ are increasing in (τ_e).

Recall that, the ends of an edge $e \in \mathbb{E}^2$ (left respectively right or bottom respectively top) are denoted by $e_x, e_y \in \mathbb{Z}^2$, and the ends of its dual edge e^* (bottom respectively top or left respectively right) are denoted by e^*_x and e^*_y. We also write $(1,0)$ for the edge with ends $(0,0), (1,0) \in \mathbb{Z}^2$.

Definition 3.6.2. *For any positive integer n, $q_1, q_2 \in [0,1]$, $z \in \mathbb{Z}^2$ and an edge $e \in B(z,n)$, we define $A_e(z;n;q_1,q_2)$ as the event that there exist 4 disjoint paths P_1–P_4 such that*

- *P_1 and P_2 are q_1-open paths in $B(z,n) \setminus \{e\}$, the path P_1 connects e_x to $\partial B(z,n)$, and the path P_2 connects e_y to $\partial B(z,n)$;*
- *P_3 and P_4 are q_2-closed paths in $B(z,n)^* \setminus \{e^*\}$, the path P_3 connects e_x^* to $\partial B(z,n)^*$, and the path P_4 connects e_y^* to $\partial B(z,n)^*$.*

We write $A_e(n;q_1,q_2)$ for $A_e(0;n;q_1,q_2)$ and $A(n;q_1,q_2)$ for $A_{(1,0)}(n;q_1,q_2)$. For any two positive integers $n < N$, $q_1, q_2 \in [0,1]$, $z \in \mathbb{Z}^2$, we define $A(z;n,N;q_1,q_2)$ as the event that there exist 4 disjoint paths, two q_1-open paths in the annulus $Ann(z;n,N)$ from $B(z,n)$ to $\partial B(z,N)$ and two q_2-closed paths in the annulus $Ann(z;n,N)^*$ from $B(z,n)^*$ to $\partial B(z,N)^*$ such that the q_1-open paths are separated by the q_2-closed paths. We write $A(n,N;q_1,q_2)$ for $A(0;n,N;q_1,q_2)$. The events $A_e(n;q_1,q_2)$ and $A(n,N;q_1,q_2)$ are illustrated in Figure 3.1.

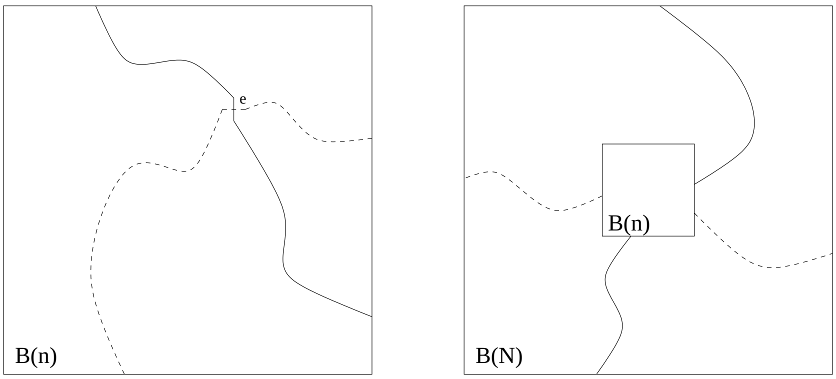

Figure 3.1: Events $A_e(n;q_1,q_2)$ and $A(n,N;q_1,q_2)$. The solid curves represent q_1-open paths, and the dotted curves represent q_2-closed paths. The edge e does not have to be q_1-open or q_2-closed.

We will follow the ideas developed in [34]. For that we need to define some subevents of $A_e(z;n;q_1,q_2)$ and $A(z;n,N;q_1,q_2)$. For $n \geq 1$, let $U_n = \partial B(n) \cap \{x_2 = n\}$, $D_n = \partial B(n) \cap \{x_2 = -n\}$, $R_n = \partial B(n) \cap \{x_1 = n\}$, $L_n = \partial B(n) \cap \{x_1 = -n\}$ be the sides of the box $B(n)$. Let $U_n(z) = z + U_n$, $D_n(z) = z + D_n$, $R_n(z) = z + R_n$, and $L_n(z) = z + L_n$ be the sides of the box $B(z,n)$.

Definition 3.6.3. *For any positive integer n, $q_1, q_2 \in [0,1]$, $z \in \mathbb{Z}^2$, and an edge $e \in B(z,n)$, we define $\bar{A}_e(z;n;q_1,q_2)$ as the event that there exist 4 disjoint paths P_1–P_4 such that*

- *P_1 and P_2 are q_1-open paths in $B(z,n) \setminus \{e\}$, the path P_1 connects e_x or e_y to $U_n(z)$, and the path P_2 connects the other end of e to $D_n(z)$;*
- *P_3 and P_4 are q_2-closed paths in $B(z,n)^* \setminus \{e^*\}$, the path P_3 connects e_x^* or e_y^* to $R_n(z)^*$, and the path P_4 connects the other end of e^* to $L_n(z)^*$.*

We define $\bar{A}_e(z;n;q_1,\cdot)$ *as the event that there exist* 2 *disjoint* q_1*-open paths* P_1 *and* P_2 *in* $B(z,n)\setminus\{e\}$*, the path* P_1 *connects* e_x *or* e_y *to* $U_n(z)$*, and the path* P_2 *connects the other end of* e *to* $D_n(z)$*.*

We write $\bar{A}_e(n;q_1,q_2)$ *for* $\bar{A}_e(0;n;q_1,q_2)$*,* $\bar{A}(n;q_1,q_2)$ *for* $\bar{A}_{(1,0)}(n;q_1,q_2)$*, and we use similar notation for the events* $A_e(z;n;q_1,\cdot)$*.*

For any two positive integers $n<N$*,* $q_1,q_2\in[0,1]$*, and* $z\in\mathbb{Z}^2$*, we define* $\bar{A}(z;n,N;q_1,q_2)$ *as the event that there exist* 4 *disjoint paths* P_1*–*P_4 *such that*

- P_1 *and* P_2 *are* q_1*-open paths in the annulus* $Ann(z;n,N)$*, the path* P_1 *connects* $U_n(z)$ *to* $U_N(z)$*, and the path* P_2 *connects* $D_n(z)$ *to* $D_N(z)$*;*
- P_3 *and* P_4 *are* q_2*-closed paths in the annulus* $Ann(z;n,N)^*$*, the path* P_3 *connects* $R_n(z)^*$ *to* $R_N(z)^*$*, and the path* P_4 *connects* $L_n(z)^*$ *to* $L_N(z)^*$*.*

We write $\bar{A}(n,N;q_1,q_2)$ *for* $\bar{A}(0;n,N;q_1,q_2)$*.*

We also need to define events similar to the events Δ in [34, Figure 8]. For any two positive integers $n<N$ and $z\in\mathbb{Z}^2$, we define $U_{n,N}(z)=z+[-n,n]\times[n+1,N]$, $D_{n,N}(z)=z+[-n,n]\times[-N,-n-1]$, $R_{n,N}(z)=z+[n+1,N]\times[-n,n]$, $L_{n,N}(z)=z+[-N,-n-1]\times[-n,n]$.

Definition 3.6.4. *For any positive integer* n*,* $q_1,q_2\in[0,1]$*,* $z\in\mathbb{Z}^2$*, and an edge* $e\in B(z,\lfloor n/2\rfloor)$*, we define* $\widetilde{A}_e(z;n;q_1,q_2)$ *as the event that*

- *the event* $\bar{A}_e(z;n;q_1,q_2)$ *occurs,*
- *the two* q_1*-open paths* P_1 *and* P_2 *from the definition of* $\bar{A}_e(z;n;q_1,q_2)$ *can be chosen such that* $P_1\cap Ann(z;\lfloor n/2\rfloor,n)\subset U_{\lfloor n/2\rfloor,n}(z)$ *and* $P_2\cap Ann(z;\lfloor n/2\rfloor,n)\subset D_{\lfloor n/2\rfloor,n}(z)$*,*
- *the two* q_2*-closed paths* P_3 *and* P_4 *from the definition of* $\bar{A}_e(z;n;q_1,q_2)$ *can be chosen such that* $P_3\cap Ann(z;\lfloor n/2\rfloor,n)^*\subset R_{\lfloor n/2\rfloor,n}(z)^*$ *and* $P_4\cap Ann(z;\lfloor n/2\rfloor,n)^*\subset L_{\lfloor n/2\rfloor,n}(z)^*$*,*
- *there exist* q_1*-open horizontal crossings of* $U_{\lfloor n/2\rfloor,n}(z)$ *and* $D_{\lfloor n/2\rfloor,n}(z)$*, and there exist* q_2*-closed vertical crossings of* $L_{\lfloor n/2\rfloor,n}(z)^*$ *and* $R_{\lfloor n/2\rfloor,n}(z)^*$*.*

We write $\widetilde{A}_e(n;q_1,q_2)$ *for* $\widetilde{A}_e(0;n;q_1,q_2)$ *and* $\widetilde{A}(n;q_1,q_2)$ *for* $\widetilde{A}_{(1,0)}(n;q_1,q_2)$*. The event* $\widetilde{A}_e(n;q_1,q_2)$ *is illustrated in Figure 3.2.*

For any positive integers n,N *such that* $4n\le N$*,* $q_1,q_2\in[0,1]$*,* $z\in\mathbb{Z}^2$*, we define* $\widetilde{A}(z;n,N;q_1,q_2)$ *as the event that*

- *the event* $\bar{A}(z;n,N;q_1,q_2)$ *occurs,*
- *the two* q_1*-open paths* P_1 *and* P_2 *from the definition of* $\bar{A}(z;n,N;q_1,q_2)$ *satisfy* $P_1\cap Ann(z;n,2n)\subset U_{n,2n}(z)$*,* $P_1\cap Ann(z;\lfloor N/2\rfloor,N)\subset U_{\lfloor N/2\rfloor,N}(z)$*,* $P_2\cap Ann(z;n,2n)\subset D_{n,2n}(z)$*, and* $P_2\cap Ann(z;\lfloor N/2\rfloor,N)\subset D_{\lfloor N/2\rfloor,N}(z)$*,*
- *the two* q_2*-closed paths* P_3 *and* P_4 *from the definition of* $\bar{A}(z;n,N;q_1,q_2)$ *satisfy* $P_3\cap Ann(z;n,2n)^*\subset R_{n,2n}(z)^*$*,* $P_3\cap Ann(z;\lfloor N/2\rfloor,N)^*\subset R_{\lfloor N/2\rfloor,N}(z)^*$*,* $P_4\cap Ann(z;n,2n)^*\subset L_{n,2n}(z)^*$*, and* $P_4\cap Ann(z;\lfloor N/2\rfloor,N)^*\subset L_{\lfloor N/2\rfloor,N}(z)^*$*,*
- *there exist* q_1*-open horizontal crossings of* $U_{n,2n}(z)$*,* $U_{\lfloor N/2\rfloor,N}(z)$*,* $D_{n,2n}(z)$ *and* $D_{\lfloor N/2\rfloor,N}(z)$*, and there exist* q_2*-closed vertical crossings of* $L_{n,2n}(z)^*$*,* $L_{\lfloor N/2\rfloor,N}(z)^*$*,* $R_{n,2n}(z)^*$ *and* $R_{\lfloor N/2\rfloor,N}(z)^*$*.*

We write $\widetilde{A}(n,N;q_1,q_2)$ *for* $\widetilde{A}(0;n,N;q_1,q_2)$*.*

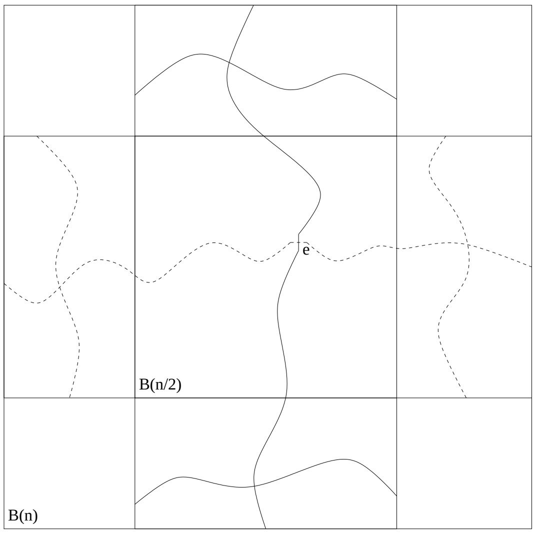

Figure 3.2: Event $\widetilde{A}_e(n; q_1, q_2)$. The solid curves represent q_1-open paths, and the dotted curves represent q_2-closed paths. The edge e does not have to be q_1-open or q_2-closed.

Lemma 3.6.5. *For any positive integers n, N such that $4n \le N$ and $q_1, q_2 \in [p_c, p_N]$,*

$$\mathbb{P}(A(n, N; q_1, q_2)) \asymp \mathbb{P}(\bar{A}(n, N; q_1, q_2)) \asymp \mathbb{P}(\widetilde{A}(n, N; q_1, q_2)), \tag{3.6.22}$$

and

$$\mathbb{P}(A(N; q_1, q_2)) \asymp \mathbb{P}(\bar{A}(N; q_1, q_2)) \asymp \mathbb{P}(\widetilde{A}(N; q_1, q_2)), \tag{3.6.23}$$

where the constants in (3.6.22) and (3.6.23) do not depend on n, N, q_1 and q_2.

Proof. The case $q_1 = q_2$ is considered in [34, Lemma 4] (see also [44, Theorem 11]). The proof is based on Lemma 3.6.1 and the RSW-Theorem. The same proof is valid for general q_1 and q_2. □

We need several corollaries of Lemma 3.6.1 and Lemma 3.6.5. Their proofs are similar to the proofs for $q_1 = q_2$ (see, e.g., Corollary 3 and Lemma 6 in [34] or Propositions 12 and 17 in [44]). We omit the details.

Corollary 3.6.6. *1. For any positive integers a, b and $n < N$ such that $an < bN$, for any $q_1, q_2 \in [p_c, p_N]$,*

$$\mathbb{P}(A(n, N; q_1, q_2)) \asymp \mathbb{P}(A(an, bN; q_1, q_2)), \tag{3.6.24}$$

where the constants in (3.6.24) only depend on a and b.

2. For any positive integers $n < m < N$ and $q_1, q_2 \in [p_c, p_N]$,

$$\mathbb{P}(A(n, N; q_1, q_2)) \asymp \mathbb{P}(A(n, m; q_1, q_2))\mathbb{P}(A(m, N; q_1, q_2)), \tag{3.6.25}$$

where the constants in (3.6.25) do not depend on n, m, N, q_1 and q_2.

3. *For any positive integer N, $q_1, q_2 \in [p_c, p_N]$ and edge $e \in B(\lfloor N/2 \rfloor)$,*

$$\mathbb{P}(A_e(N; q_1, q_2)) \asymp \mathbb{P}(\bar{A}_e(N; q_1, q_2)) \asymp \mathbb{P}(\widetilde{A}_e(N; q_1, q_2)) \asymp \mathbb{P}(A(N; q_1, q_2)), \tag{3.6.26}$$

where the constants in (3.6.26) do not depend on N, q_1, q_2 and e.

The proof of the lower bound in Theorem 3.1.6 is based on the following lemma.

Lemma 3.6.7. *For any positive integer N, $q_1, q_2 \in [p_c, p_N]$ and $e \in B(\lfloor N/2 \rfloor)$,*

$$\mathbb{P}(A_e(N; q_1, q_2)) \asymp \mathbb{P}(A(N; p_c, p_c)), \tag{3.6.27}$$

where the constants in (3.6.27) do not depend on N, q_1, q_2 and e.

Proof of Lemma 3.6.7. The proof for $q_1 = q_2$ is given in [34, Lemma 8]; see also [44, Theorem 27] for a detailed proof. In this case the probability measure $\mathbb{P}$ can be replaced with the probability measure $\mathbb{P}_{q_1}$ on configurations of open and closed edges. This is not the case when $q_1 \neq q_2$, which makes the proof of (3.6.27) more involved. Note that by (3.6.23) and (3.6.26) it is sufficient to show that, for $q_1, q_2 \in [p_c, p_N]$,

$$\mathbb{P}(\bar{A}(N; q_1, q_2)) \asymp \mathbb{P}(\bar{A}(N; p_c, p_c)).$$

It is immediate from monotonicity in q_1 and q_2 that

$$\mathbb{P}(\bar{A}(N; p_c, q_2)) \leq \mathbb{P}(\bar{A}(N; q_1, q_2)) \leq \mathbb{P}(\bar{A}(N; q_1, p_c)).$$

Therefore it remains to show that there exist constants D_1 and D_2 so that, for all $q_1, q_2 \in [p_c, p_N]$,

$$\mathbb{P}(\bar{A}(N; p_c, q_2)) \geq D_1 \mathbb{P}(\bar{A}(N; p_c, p_c)) \text{ and } \mathbb{P}(\bar{A}(N; q_1, p_c)) \leq D_2 \mathbb{P}(\bar{A}(N; p_c, p_c)).$$

Since the proofs of the above inequalities are similar, we only prove the first inequality. For that we use a generalization of Russo's formula [26]. We take a small $\delta > 0$. The difference $\mathbb{P}(\bar{A}(N; p_c, p)) - \mathbb{P}(\bar{A}(N; p_c, p+\delta))$ can be written as the sum

$$\delta \sum_{e \in B(N),\ e \neq (1,0)} \mathbb{P}(\bar{A}(N; p_c, \cdot), \bar{A}_e(N; p, \cdot), D_e(N; p)) + O(\delta^2),$$

where $D_e(N; p)$ is the event that there exist three p-closed paths $P_1 - P_3$ in $B(N)^*$; the path P_1 connects an end of the edge $(1,0)^*$ to an end of the edge e^*; the path P_2 connects the other end of the edge $(1,0)^*$ to R_N^*, and the path P_3 connects the other end of the edge e^* to L_N^*; or the path P_2 connects the other end of the edge $(1,0)^*$ to L_N^*, and the path P_3 connects the other end of the edge e^* to R_N^*. Letting δ tend to 0, we obtain

$$\frac{d}{dp} \mathbb{P}(\bar{A}(N; p_c, p)) = -\sum_e \mathbb{P}(\bar{A}(N; p_c, \cdot), \bar{A}_e(N; p, \cdot), D_e(N; p)). \tag{3.6.28}$$

We write the r.h.s. of (3.6.28) as

$$- \sum_{j=1}^{\lfloor N/2 \rfloor} \sum_{e\,:\,|e_x|=j} \mathbb{P}(\bar{A}(N; p_c, \cdot), \bar{A}_e(N; p, \cdot), D_e(N; p)) \tag{3.6.29}$$

$$- \sum_{j=\lfloor N/2 \rfloor+1}^{N} \sum_{e\,:\,|e_x|=j} \mathbb{P}(\bar{A}(N; p_c, \cdot), \bar{A}_e(N; p, \cdot), D_e(N; p)). \tag{3.6.30}$$

By independence, the sum (3.6.29) is bounded from below by

$$- \sum_{j=1}^{\lfloor N/2 \rfloor} \sum_{e\,:\,|e_x|=j} \mathbb{P}(A(\lfloor j/2 \rfloor; p_c, p))\mathbb{P}(A(\lfloor 3j/2 \rfloor, N; p_c, p))\mathbb{P}(A_e(e_x; \lfloor j/2 \rfloor; p, p)).$$

We use (3.6.24), the bound $\sharp\{e \ : \ |e_x| = j\} \leq 16j$, and the fact that Lemma 3.6.7 is proven for $q_1 = q_2$ to bound the above sums from below by

$$\begin{aligned} & -C_1 \sum_{j=1}^{\lfloor N/2 \rfloor} j\mathbb{P}(A(j; p_c, p))\mathbb{P}(A(j, N; p_c, p))\mathbb{P}(A(j; p_c, p_c)) \\ \geq \quad & -C_2\mathbb{P}(A(N; p_c, p)) \sum_{j=1}^{\lfloor N/2 \rfloor} j\mathbb{P}(A(j; p_c, p_c)), \end{aligned} \tag{3.6.31}$$

where the inequality follows from (3.6.25). We estimate the sum in (3.6.31) using the relation

$$\sum_{j=1}^{N} j\mathbb{P}(A(j; p_c, p_c)) \asymp N^2\mathbb{P}(A(N; p_c, p_c)). \tag{3.6.32}$$

The relation (3.6.32) follows from (3.6.25) and the fact that $\mathbb{P}(A(j, N; p_c, p_c)) \geq C_3(j/N)^{2-C_4}$ for some positive C_3 and C_4 that do not depend on j and N. This fact follows, for example, from [44, Theorem 24], where the 5-arms exponent is computed for site percolation on the triangular lattice. The same proof applies to bond percolation on the square lattice.

Similarly to the proof of [50, Lemma 6.2], the sum (3.6.30) can be bounded from below by

$$-C_5N^2\mathbb{P}(A(N; p_c, p))\mathbb{P}(A(N; p_c, p_c)).$$

This follows from a priori estimates of probabilities of two arms in a half-plane. We refer the reader to the proof of [50, Lemma 6.2] for more details. Again, although the proof of [50, Lemma 6.2] is given for site percolation on the triangular lattice, it applies to bond percolation on the square lattice too.

Putting together the bounds for the sums (3.6.29) and (3.6.30) and using (3.6.23), we obtain that the r.h.s. of (3.6.28) is bounded from below by

$$-C_6N^2\mathbb{P}(\bar{A}(N; p_c, p))\mathbb{P}(A(N; p_c, p_c)).$$

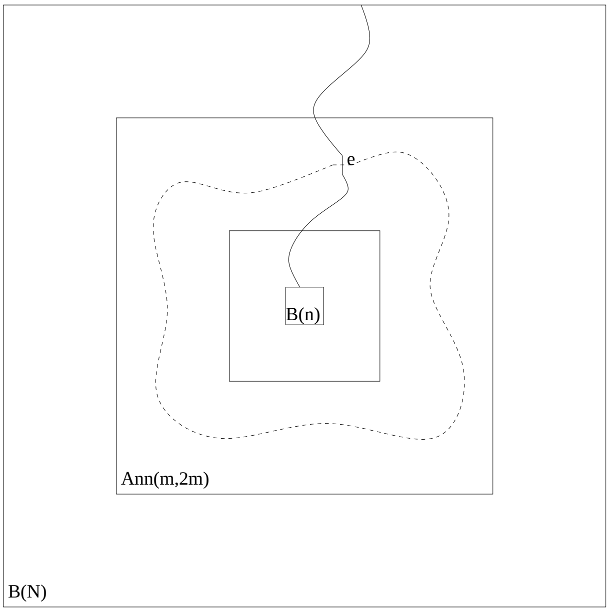

Figure 3.3: Event $C_e(n, N; m)$. The solid curves represent p_c-open paths, and the dotted curves represent p_m-closed paths. The edge e does not have to be p_c-open or p_m-closed.

Therefore,

$$\frac{d}{dp} \log \mathbb{P}(\bar{A}(N; p_c, p)) \geq -C_6 N^2 \mathbb{P}(A(N; p_c, p_c)), \tag{3.6.33}$$

and

$$\begin{aligned}
\mathbb{P}(\bar{A}(N; p_c, p)) &\geq \mathbb{P}(\bar{A}(N; p_c, p_c)) e^{-C_6 (p - p_c) N^2 \mathbb{P}(A(N; p_c, p_c))} \\
&\geq \mathbb{P}(\bar{A}(N; p_c, p_c)) e^{-C_6 (p_N - p_c) N^2 \mathbb{P}(A(N; p_c, p_c))} \\
&\geq C_7 \mathbb{P}(\bar{A}(N; p_c, p_c)).
\end{aligned}$$

In the last inequality we use (3.2.10). □

Definition 3.6.8. *For any positive integers $n \leq m \leq 2m \leq N$ and edge $e \in Ann(m, 2m)$, we define $C_e(n, N; m)$ as the event that*

- *there exist two disjoint p_c-open paths P_1 and P_2 inside $Ann(n, N) \setminus \{e\}$, the path P_1 connects e_x or e_y to $B(n)$, and the path P_2 connects the other end of e to $\partial B(N)$; and*
- *there exists a p_m-closed path P connecting e_x^* and e_y^* inside $Ann(m, 2m)^* \setminus \{e^*\}$ so that $P \cup \{e^*\}$ is a circuit around the origin in $Ann(m, 2m)^*$.*

Note that if event $C_e(n, N; m) \cap \{\tau_e \in (p_c, p_m)\}$ occurs then there is no p_c-open crossing of $Ann(n, N)$ and no p_m-closed circuit in $Ann(m, 2m)^*$ (see Figure 3.3).

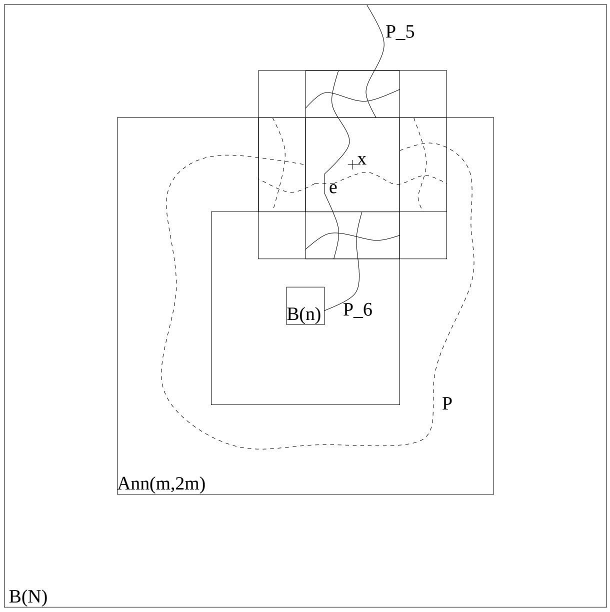

Figure 3.4: Event $\widetilde{C}_e(n, N; m)$. The solid curves represent p_c-open paths, and the dotted curves represent p_m-closed paths. The edge e does not have to be p_c-open or p_m-closed.

Definition 3.6.9. *Let n, m and N be positive integers such that $2n \le m$ and $3m \le N$. Let $x = (\lfloor m/2 \rfloor, \lfloor 3m/2 \rfloor)$. For $e \in B(x, \lfloor m/2 \rfloor)$, we define $\widetilde{C}_e(n, N; m)$ as the event that*

- *the event $\widetilde{A}_e(x; m; p_c, p_m)$ occurs;*
- *there are two disjoint p_c-open paths P_5 and P_6, so that P_5 connects $U_{\lfloor m/2 \rfloor}(x)$ to the boundary of $B(N)$ inside $Ann(2m-1, N)$, and P_6 connects $D_{\lfloor m/2 \rfloor}(x)$ to the boundary of $B(n)$ inside $Ann(n, m)$. Moreover, P_5 and P_6 satisfy $P_5 \cap Ann(x; \lfloor m/2 \rfloor, m) \subset U_{\lfloor m/2 \rfloor, m}(x)$ and $P_6 \cap Ann(x; \lfloor m/2 \rfloor, m) \subset D_{\lfloor m/2 \rfloor, m}(x)$;*
- *there exists a p_m-closed path P inside $Ann(m, 2m-1)^* \setminus B(x, \lfloor m/2 \rfloor)^*$ so that P connects $L_{\lfloor m/2 \rfloor}(x)^*$ to $R_{\lfloor m/2 \rfloor}(x)^*$ and $P \cap Ann(x; \lfloor m/2 \rfloor, m)^* \subset L_{\lfloor m/2 \rfloor, m}(x)^* \cup R_{\lfloor m/2 \rfloor, m}(x)^*$.*

The event $\widetilde{C}_e(n, N; m)$ is illustrated in Figure 3.4.

The event $\widetilde{C}_e(n, N; m)$ obviously implies the event $C_e(n, N; m)$. The reason we introduce the event $\widetilde{C}_e(n, N; m)$ is that

$$\mathbb{P}(\widetilde{C}_e(n, N; m)) \asymp \mathbb{P}(\widetilde{A}_e(x; m; p_c, p_m))\mathbb{P}_{cr}(B(n) \leftrightarrow \partial B(N)), \qquad (3.6.34)$$

where the constants do not depend on e, m, n and N. This observation follows from Lemma 3.6.1, RSW-Theorem, and (3.6.24) and (3.6.25) applied to $q_1 = q_2 = p_c$.

Corollary 3.6.10. *For any positive integers n, m and N such that $2n \leq m$ and $3m \leq N$,*

$$\mathbb{P}(\exists e \in Ann(m, 2m) \ : \ \tau_e \in (p_c, p_m), C_e(n, N; m)) \geq C_8 \mathbb{P}_{cr}(B(n) \leftrightarrow \partial B(N)), \tag{3.6.35}$$

where C_8 does not depend on n, N, and m.

Proof of Corollary 3.6.10. Note that the events

$$\{\tau_e \in (p_c, p_m), C_e(n, N; m)\}_{e \in Ann(m,2m)}$$

are disjoint. Therefore,

$$\begin{aligned}
&\mathbb{P}(\exists e \in Ann(m, 2m) \ : \ \tau_e \in (p_c, p_m), C_e(n, N; m)) \\
= & \sum_{e \in Ann(m,2m)} \mathbb{P}(\tau_e \in (p_c, p_m), C_e(n, N; m)) \\
\geq & \ (p_m - p_c) \sum_{e \in B(x, \lfloor m/2 \rfloor)} \mathbb{P}(\widetilde{C}_e(n, N; m)) \\
\geq & \ C_9 (p_m - p_c) \sum_{e \in B(x, \lfloor m/2 \rfloor)} \mathbb{P}(\widetilde{A}_e(x; m; p_c, p_m)) \mathbb{P}_{cr}(B(n) \leftrightarrow \partial B(N)) \\
\geq & \ C_{10} (p_m - p_c) m^2 \mathbb{P}(A(m; p_c, p_c)) \mathbb{P}_{cr}(B(n) \leftrightarrow \partial B(N)) \\
\geq & \ C_{11} \mathbb{P}_{cr}(B(n) \leftrightarrow \partial B(N)).
\end{aligned}$$

The second inequality follows from (3.6.34). In the third inequality we use (3.6.26) and Lemma 3.6.7. In the last inequality we use (3.2.10). □

Proof of Theorem 3.1.6. Lower bound. We give the proof for $k = 2$. The case $k = 1$ is proven above, and the proof for $k \geq 3$ is similar to the one for $k = 2$. Note that event $\{\widehat{R}_2 > n\}$ is implied by the event that there exists an edge $e \in B(n)$ and $p > p_c$ such that

- $\tau_e \in (p_c, p)$;
- there exist two p_c-open paths P_1 and P_2 in $B(n)$, the path P_1 connects an end of e to the origin, and the path P_2 connects the other end of e to the boundary of $B(n)$;
- there exists a p-closed path P in $B(n)^*$ connecting e_x^* to e_y^* so that $P \cup \{e^*\}$ is a circuit around the origin.

There could be at most one edge $e \in B(n)$ which satisfies the above three conditions. Therefore,

$$\begin{aligned}
\mathbb{P}(\widehat{R}_2 > n) \ \geq & \sum_{k=0}^{\lfloor \log n \rfloor - 1} \mathbb{P}(\exists e \in Ann(\lfloor n/2^{k+1} \rfloor, \lfloor n/2^k \rfloor) \ : \ \tau_e \in (p_c, p_{\frac{n}{2^{k+1}}}), C_e(1, n; \lfloor n/2^{k+1} \rfloor)) \\
\geq & \ C_{12} \sum_{k=0}^{\lfloor \log n \rfloor - 1} \mathbb{P}_{cr}(0 \leftrightarrow \partial B(n)) \\
= & \ C_{12} \lfloor \log n \rfloor \mathbb{P}_{cr}(0 \leftrightarrow \partial B(n)).
\end{aligned}$$

The last inequality follows from (3.6.35). □

3.6.3 Proof of Theorem 3.1.10

For short, we use the following notation:

$$\begin{aligned}\pi(n) &= P_{cr}(0 \leftrightarrow \partial B(n)),\\ \pi(n,p) &= P_p(0 \leftrightarrow \partial B(n)),\\ s(n) &= n^2\pi(n).\end{aligned}$$

The difficult part of Theorem 3.1.10 is the upper bound. We need the following key ingredient.

Lemma 3.6.11. *There exist constants C_1 and C_2, such that*

$$\begin{aligned}&P_p\Big(0 \leftrightarrow \infty,\, |\mathcal{C}(0)\cap B(2^k)| > s(n)\Big)\\ &\qquad \le \theta(p)\, 2C_1\, \exp\left\{-(2C_2)^{-1}\frac{s(n)}{2^{2k}\pi(2^k,p)}\right\}, \quad p > p_c, 2^k \le n.\end{aligned}$$

Proof. The proof is based on the following moment estimate:

$$E_p\Big(|\mathcal{C}(0)\cap B(2^k)|^t\Big|0 \longleftrightarrow \infty\Big) \le C_1\, t!\, \Big[C_2\, 2^{2k}\, \pi(2^k,p)\Big]^t, \quad t \ge 1. \quad (3.6.36)$$

Very similar estimates were proved in [33, Theorem (8)] and in [43]. To adapt their proofs in order to obtain (3.6.36), one merely needs that the inequality $\sum_{m=0}^n \pi(m,p) \le Cn\pi(n,p)$ holds for all $p \ge p_c$ (with some constant C independent of p). From (3.6.36), we readily get

$$E_p\left(\exp\left\{\lambda\frac{|\mathcal{C}(0)\cap B(2^k)|}{2^{2k}\,\pi(2^k,p)}\right\}\,\middle|\, 0 \longleftrightarrow \infty\right) \le C_1\frac{1}{1-\lambda C_2}, \quad 0 < \lambda < C_2^{-1}.$$

Taking $\lambda = (2C_2)^{-1}$ we easily obtain the estimate of the lemma. □

Proof of Theorem 3.1.10. We prove only the $k=1$ case. The proof for other ponds is rather lengthy and very similar to the proof Theorem 3.1.6. The first inequality follows from [33, Remark (9)]. The second inequality follows immediately from the fact that the p_c-open cluster containing the origin is a subset of $\hat{V}_1$.

The third inequality will be proved by a decomposition, somewhat similar to the one in Theorem 3.1.6, but now two-fold: this time we will also decompose according to the value of $\hat{R}_1$. As in the proof of Theorem 3.1.6, without loss of generality we may assume that n is of the form 2^N.

Let

$$E_{n,k} = \{2^{k-1} < \hat{R}_1 \le 2^k,\, |\hat{V}_1| > s(n)\}.$$

Note that $s(n) \ge C_3 n^{3/2}$, and $|B(2^k)| \le C_4 2^{2k}$. Letting

$$k_0 := \max\{k : C_4 2^{2k} \le C_3 n^{3/2}\},$$

for $k < k_0$, $\hat{R}_1 \leq 2^k$ implies $|\hat{V}_1| \leq C_4 2^{2k} \leq s(n)$, and hence $E_{n,k} = \emptyset$. Therefore, we can write

$$P(|\hat{V}_1| > s(n)) \leq P(\hat{R}_1 > n) + \sum_{k=k_0}^{N} P(E_{n,k}). \tag{3.6.37}$$

The first term on the right hand side is at most $C_5\pi(n)$, by Theorem 3.1.6. Consider now a general term of the sum. We decompose this according to the value of $\hat{\tau}$ as follows:

$$\begin{aligned} &P(2^{k-1} < \hat{R}_1 \leq 2^k,\, |\hat{V}_1| > s(n)) \\ &\quad = P(E_{n,k},\, \hat{\tau} > p_k(0)) + \sum_{j=0}^{\log^* k} P(E_{n,k},\, p_k(j+1) < \hat{\tau} < p_k(j)), \end{aligned} \tag{3.6.38}$$

where we let $p_k(\log^* k + 1) = p_c$.

We first look at the event in the first term on the right hand side. This event implies the occurrence of $A_{2^{k-1},p_k(0)}$. Hence, by virtue of (3.6.5), its probability is at most $C_6(2^{2k})^{-\tilde{C}_4 C_2}$. By the choice of C_2, we have $\tilde{C}_4 C_2 \geq 1$. Hence the sum over $k_0 \leq k \leq N$ is bounded by $C_7(2^{2k_0})^{-1}$. By the definition of k_0, this is $o(\pi(n))$.

Consider now the event in the general term on the right hand side of (3.6.38). This event implies the following two events:

(i) $A_{2^{k-1},p_k(j+1)}$;

(ii) $\{0 \xleftrightarrow{p_k(j)} \infty,\, |\mathcal{C}(0; p_k(j)) \cap B(2^k)| > s(n)\}$;

where $\mathcal{C}(0;p)$ denotes the p-open cluster of 0. Since (i) is a decreasing and (ii) an increasing event, the Harris-FKG inequality yields that the general term in (3.6.38) is at most the product of the probabilities of event (i) and event (ii).

As to event (i), the same arguments that led to (3.6.8) (and noting the Remark a few lines below (3.6.5)) show that for $j < \log^* k$ this has probability less than or equal to

$$C_8(\log^{(j)} k)^{-1} \tag{3.6.39}$$

It is easy to see that, after increasing the value of C_8 if necessary, this bound even holds for $j = \log^* k$.

As to event (ii), by Lemma 3.6.11 this has probability at most

$$\theta(p_k(j))(2C_1)\exp\left\{-(2C_2)^{-1}\frac{s(n)}{2^{2k}\,\pi(2^k, p_k(j))}\right\}. \tag{3.6.40}$$

Applying the first inequality in (3.2.6) to the probability in the exponent in (3.6.40), and then applying (3.6.7) twice, shows that (3.6.40) is at most a constant times

$$\pi(2^k)(\log^{(j)} k)^{1/2}\exp\left\{-C_{10}\frac{2^{2N}\pi(n)}{2^{2k}\pi(2^k)}(\log^{(j)} k)^{-1/2}\right\}. \tag{3.6.41}$$

Combining this with (3.6.39) gives that the general term in (3.6.38) is at most

$$C_9\pi(n)(\log^{(j)} k)^{-1/2}\frac{\pi(2^k)}{\pi(n)}\exp\left\{-C_{10}\frac{2^{2N}\pi(n)}{2^{2k}\pi(2^k)}(\log^{(j)} k)^{-1/2}\right\}. \qquad (3.6.42)$$

Due to (3.2.8), this as at most

$$C_{11}\pi(n)(\log^{(j)} k)^{-1/2}2^{(N-k)/2}\exp\left\{-C_{12}2^{(N-k)(3/2)}(\log^{(j)} k)^{-1/2}\right\}. \qquad (3.6.43)$$

We split the sums over j and k into two parts:

(1) $2^{(N-k)} \le (\log^{(j)} k)^{1/2}$;

(2) $2^{(N-k)} > (\log^{(j)} k)^{1/2}$.

In case (1), we bound the exponential in (3.6.43) by 1, and we have

$$(\log^{(j)} k)^{-1/2}2^{(N-k)/2} \le (\log^{(j)} k)^{-1/4} \le C_{13}(\log^{(j)} N)^{-1/4}.$$

The number of possible values of k is at most

$$(2\log 2)^{-1}\log^{(j+1)} k \le C_{14}(\log^{(j)} N)^{1/8}.$$

Hence the contribution of this case is bounded by

$$\sum_{j=0}^{\log^* N}(\log^{(j)} N)^{-1/8} \le C_{15}.$$

In case (2), we bound the exponential by $\exp\{-C_{12}2^{(N-k)/2}\}$, and we have $(\log^{(j)} k)^{-1/2}2^{(N-k)/2} \le 2^{(N-k)/2}$. The sum over k can be bounded as follows:

$$\sum_{k:N-k\ge c\log^{(j+1)} N} 2^{(N-k)/2}\exp\{-C_{12}2^{(N-k)/2}\} \le C_{16}\exp\{-C_{17}(\log^{(j)} N)^{c_1}\},$$

for some $c_1 > 0$. The sum of the right hand side over j is again bounded. This proves the theorem. □

3.7 Proof of Theorem 3.1.12

Let $G = (\mathcal{G}, \mathcal{E})$ be an infinite connected subgraph of $(\mathbb{Z}^2, \mathbb{E}^2)$ which contains the origin. We call an edge $e \in \mathcal{E}$ a *disconnecting edge* for G if the graph $(\mathcal{G}, \mathcal{E}\setminus\{e\})$ has a finite component, and if the origin belongs to this finite component. Note that each outlet of the invasion is a disconnecting edge for the IPC.

Let $D_{m,n}$ be the event that the IIC does not contain a disconnecting edge in the annulus $Ann(m,n)$, and let $\mathcal{D}_{m,n}$ be the event that the IPC does not contain a disconnecting edge in the annulus $Ann(m,n)$. We prove the following theorem:

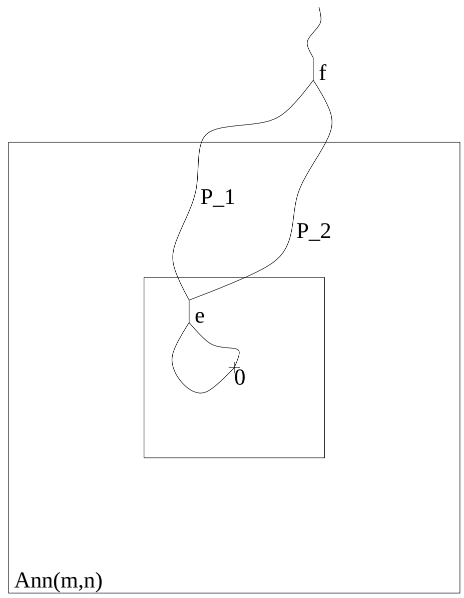

Figure 3.5: Event $\mathcal{D}_{m,n}$. The edges e and f are disconnecting. The paths P_1 and P_2 create a circuit, which implies that there is a pond that entirely contains both paths.

Theorem 3.7.1. *There exists a sequence* (n_k) *such that*

$$\mathbb{P}(\sum_k \mathbb{I}(\mathcal{D}_{n_k,n_{k+1}}) < \infty) = 1, \tag{3.7.1}$$

and

$$\nu(\sum_k \mathbb{I}(D_{n_k,n_{k+1}}) = \infty) = 1. \tag{3.7.2}$$

Theorem 3.1.12 immediately follows from Theorem 3.7.1. Indeed, Theorem 3.7.1 implies that IIC is supported on clusters for which infinitely many of the events $D_{n_k,n_{k+1}}$ occur, and IPC is supported on clusters for which only finitely many of the events $\mathcal{D}_{n_k,n_{k+1}}$ occur. Roughly speaking, this says that the distance between consecutive disconnecting edges (ordered by distance from the origin) can be much larger in the IIC than in the IPC. The proof of Theorem 3.7.1 is based on the following result (see Section 3.1.2 for the definitions).

Theorem 3.7.2. *There exists* C_1, C_2 *such that, for all* $1 \le m < n$,

$$\mathbb{P}(\mathcal{D}_{m,n}) \le C_1 \mathbb{P}_{cr}(A^2_{m,n}); \tag{3.7.3}$$

and

$$\nu(D_{m,n}) \ge C_2 \frac{\mathbb{P}_{cr}(A^2_{m,n})}{\mathbb{P}_{cr}(A^1_{m,n})}. \tag{3.7.4}$$

Lemma 3.7.3. *[44, Theorem 27] For all positive integers $m < n$ and for all $p \in [p_c, p_n]$,*

$$\mathbb{P}_p(A^2_{m,n}) \asymp \mathbb{P}_{cr}(A^2_{m,n}),$$

where the constants do not depend on m, n and p.
Although Theorem 27 in [44] is stated for site percolation on the triangular lattice, the proof for bond percolation on the square lattice is the same.

Lemma 3.7.4. *There exists C_3 so that for all $m_1 < m_2 < n$ we have*

$$\frac{\mathbb{P}_{cr}(A^2_{m_1,n})}{\mathbb{P}_{cr}(A^2_{m_1,m_2})} \geq C_3 \frac{m_2}{n}.$$

Proof. This follows from a priori estimates of probabilities of two arms in a half-plane (see [44, Theorem 24]). □

Proof of Theorem 3.7.2. We first prove (3.7.3). Note that if the invasion percolation cluster contains a circuit then there is a pond that entirely contains this circuit. Therefore, the event $\mathcal{D}_{m,n}$ can only occur if there exists an invasion pond which contains two disjoint crossings P_1 and P_2 of the annulus $Ann(m,n)$ (see Figure 3.5). Therefore there exists p' such that P_1 and P_2 are p'-open and there exists a circuit around the origin which is p'-closed and which has diameter at least n.

Recall the definition of $(p_n(j))$ from (3.2.3). Later we take C_* in (3.2.3) large enough. We decompose the event $\mathcal{D}_{m,n}$ according to the value of p':

$$\mathbb{P}(\mathcal{D}_{m,n}) = \sum_{j=1}^{\log^* n} \mathbb{P}(\mathcal{D}_{m,n}; p' \in [p_n(j), p_n(j-1))). \tag{3.7.5}$$

Note that the event $\{\mathcal{D}_{m,n}; p' \in [p_n(j), p_n(j-1))\}$ implies the event $A^2_{m,n,p_n(j-1)} \cap B_{n,p_n(j)}$ (see Section 3.1.2 for the definition of these events). It follows from (3.2.5) and (3.2.9) that there exist constants C_4 and C_5 such that the probability $\mathbb{P}(B_{n,p_n(j)})$ is bounded from above by $C_4(\log^{(j-1)} n)^{-C_5}$. The constant C_5 can be made arbitrarily large by making C_* large enough. We use Lemma 3.7.3 and Lemma 3.7.4 to bound the probability $\mathbb{P}(A^2_{m,n,p_n(j-1)}) \leq C_6(\log^{(j-1)} n)\mathbb{P}_{cr}(A^2_{m,n})$. We use the FKG inequality and the above estimates for the events $A^2_{m,n,p_n(j-1)}$ and $B_{n,p_n(j)}$ to get

$$\mathbb{P}(\mathcal{D}_{m,n}) \leq C_4 C_6 \mathbb{P}_{cr}(A^2_{m,n}) \sum_{j=1}^{\log^* n} (\log^{(j-1)} n)^{1-C_5} \leq C_7 \mathbb{P}_{cr}(A^2_{m,n}).$$

The last inequality follows from [30, (2.26)] if we take C_* so that $C_5 > 1$.

We now prove (3.7.4). Let $C_{m,n}$ be the event $A_{\lfloor m/2 \rfloor,m} \cap A^2_{\lfloor m/2 \rfloor,2n} \cap A_{n,2n}$ (see Figure 3.6). Note that $\nu(C_{m,n} \setminus D_{m,n}) = 0$. Therefore it is sufficient to prove (3.7.4) for $C_{m,n}$.

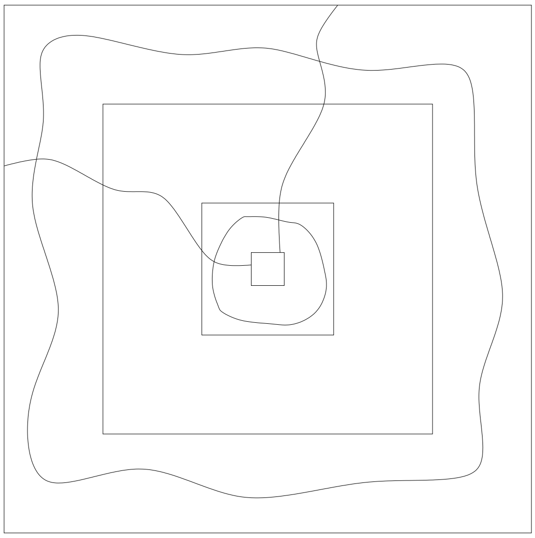

Figure 3.6: Event $C_{m,n}$. The inner circuit is in the annulus $Ann(\lfloor m/2\rfloor, m)$, and the outer circuit is in the annulus $Ann(n, 2n)$.

For positive integers $m < n < N$ (later we consider the limit as N tends to infinity), we use the FKG inequality to get

$$\begin{aligned}\mathbb{P}_{cr}(C_{m,n}\cap A^1_{0,N}) &\geq \mathbb{P}_{cr}(C_{m,n}\cap A^1_{0,m}\cap A^1_{n,N})\\ &\geq C_8\mathbb{P}_{cr}(A^2_{\lfloor m/2\rfloor,2n})\mathbb{P}_{cr}(A^1_{0,m})\mathbb{P}_{cr}(A^1_{n,N})\\ &\geq C_8\frac{\mathbb{P}_{cr}(A^2_{\lfloor m/2\rfloor,2n})\mathbb{P}_{cr}(A^1_{0,N})}{\mathbb{P}_{cr}(A^1_{m,n})}\end{aligned}$$

for some $C_8 > 0$. Standard RSW arguments give a constant C_9 so that for all $1 \leq m < n$,

$$\mathbb{P}_{cr}(A^2_{\lfloor m/2\rfloor,2n}) \geq C_9\mathbb{P}_{cr}(A^2_{m,n}).$$

Therefore

$$\nu(D_{m,n}) \geq C_8C_9\frac{\mathbb{P}_{cr}(A^2_{m,n})}{\mathbb{P}_{cr}(A^1_{m,n})}.$$

□

Proposition 3.7.5. *There exists a sequence (n_k) such that $n_{k+1} > 4n_k$,*

$$\sum_k \mathbb{P}_{cr}(A^2_{n_k,n_{k+1}}) < \infty, \tag{3.7.6}$$

and

$$\sum_k \frac{\mathbb{P}_{cr}(A^2_{n_{2k},n_{2k+1}})}{\mathbb{P}_{cr}(A^1_{n_{2k},n_{2k+1}})} = \infty. \tag{3.7.7}$$

Proposition 3.7.5 follows from Lemma 3.7.4 and the fact that $\mathbb{P}_{cr}(A^1_{m,n}) \le c(m/n)^\delta$ for some positive c and δ. Indeed, we obtain $\mathbb{P}_{cr}(A^1_{m,n}) \le c(C_3\mathbb{P}_{cr}(A^2_{m,n}))^\delta$. We now take, for example, the sequence $n_k = \min\{n > 4n_{k-1} \; : \; \mathbb{P}_{cr}(A^2_{n_{k-1},n}) \le (1/k)^{1+\delta}\}$.

Proof of Theorem 3.7.1. We take a sequence from Proposition 3.7.5. Equality (3.7.1) follows from Borel-Cantelli's lemma. To prove (3.7.2), we use Borel's lemma [39]:

Lemma 3.7.6. *Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sequence of events $\Gamma_n \in \mathcal{F}$. Let $\limsup_n \Gamma_n = \cap_n \cup_{k\ge n}\Gamma_k$ be the event that infinitely many of Γ_n's occur. Let $a_n = \mathbb{I}(\Gamma_n)$ be the indicator of event Γ_n. If there exists a sequence b_n such that $\sum_n b_n = \infty$ and for any $\alpha_i \in \{0,1\}$, $i = 1, \ldots, n-1$,*

$$\mathbb{P}(\Gamma_n \mid a_1 = \alpha_1, \ldots, a_{n-1} = \alpha_{n-1}) \ge b_n > 0$$

then

$$\mathbb{P}(\limsup_n \Gamma_n) = 1.$$

Note that it is sufficient to prove (3.7.2) for the events $C_{n_k,n_{k+1}}$ (see the proof of Theorem 3.7.2 for the definition). We apply Lemma 3.7.6 to the probability measure ν and to the events $C_{n_{2k},n_{2k+1}}$. Let $d_k = \mathbb{I}(C_{n_{2k},n_{2k+1}})$. A slight extension of the proof of (3.7.4) gives for any $\alpha_i \in \{0,1\}$, $i = 1, \ldots, k-1$,

$$\nu(C_{n_{2k},n_{2k+1}} \mid d_1 = \alpha_1, \ldots, d_{k-1} = \alpha_{k-1}) \ge C_2 \frac{\mathbb{P}_{cr}(A^2_{n_{2k},n_{2k+1}})}{\mathbb{P}_{cr}(A^1_{n_{2k},n_{2k+1}})} =: b_k, \quad (3.7.8)$$

where C_2 is the constant from (3.7.4). Indeed, let $\mathcal{W}$ be the set of configurations of edges in $B(2n_{2k-1})$ such that $d_1 = \alpha_1, \ldots, d_{k-1} = \alpha_{k-1}$. For any $\omega \in \mathcal{W}$ and large enough N,

$$\begin{aligned}
\mathbb{P}_{cr}(C_{n_{2k},n_{2k+1}} \cap A^1_{0,N} \mid \omega) &\ge \mathbb{P}_{cr}(C_{n_{2k},n_{2k+1}} \cap A^1_{0,n_{2k}} \cap A^1_{n_{2k+1},N} \mid \omega) \\
&\ge C_8 \mathbb{P}_{cr}(A^2_{\lfloor n_{2k}/2 \rfloor, 2n_{2k+1}} \mid \omega) \mathbb{P}_{cr}(A^1_{0,n_{2k}} \mid \omega) \\
&\quad\; \mathbb{P}_{cr}(A^1_{n_{2k+1},N} \mid \omega) \\
&= C_8 \mathbb{P}_{cr}(A^2_{\lfloor n_{2k}/2 \rfloor, 2n_{2k+1}}) \mathbb{P}_{cr}(A^1_{0,n_{2k}} \mid \omega) \\
&\quad\; \mathbb{P}_{cr}(A^1_{n_{2k+1},N}) \\
&\ge C_8 \frac{\mathbb{P}_{cr}(A^2_{\lfloor n_{2k}/2 \rfloor, 2n_{2k+1}}) \mathbb{P}_{cr}(A^1_{0,N} \mid \omega)}{\mathbb{P}_{cr}(A^1_{n_{2k},n_{2k+1}})},
\end{aligned}$$

which implies (3.7.8). In the second line we use the FKG inequality and independence. The equality follows from independence. From the choice of (n_k), it follows that $\sum_k b_k = \infty$. Therefore, equality (3.7.2) follows from Lemma 3.7.6. □

Chapter 4

A Stochastic Dynamics for the Incipient Infinite Cluster

4.1 Introduction

When studying the relations between invasion percolation and critical bond percolation in the plane we observed several similarities and differences between the invasion percolation cluster (IPC) and the incipient infinite cluster (IIC), introduced in [33]. In particular, we have shown that the distributions of the IPC and the IIC are mutually singular. This result was first obtained in [4] for the regular tree where, among other things, it is proved that the IIC measure stochastically dominates the IPC measure. After finding a proof of the mutual singularity in the plane a natural next step is to investigate the question of stochastic domination. The proof for regular trees in [4] uses a coupling argument that, unfortunately, relies heavily on the special structure of the underlying graph and it cannot modified to work on $\mathbb{Z}^2$. For this reason we started our attempts with trying to find an appropriate coupling through dynamic processes, namely constructing Markov chains that have the IIC and the IPC measures as stationary distributions and that can be used to find a proper coupling. This leads to the introduction of the three different models in the next section. The precise definition will be given in the next section but the underlying idea is the same in all three cases: we choose an initial configuration on $\mathbb{Z}^2$ such that the origin is in an infinite cluster. We attach a Poisson clock with certain rate to each bond in $\mathbb{Z}^2$, independently of each other. When the clock of a certain bond rings we try to update the state of the edge. We first check whether the edge is pivotal for the origin being connected to infinity, i.e. whether changing the state of the edge from open to closed will disconnect the origin from infinity. If the edge is found to be pivotal, we do not change its state. Note that in this case the edge is necessarily open. If, on the other hand, the bond is not pivotal we make it open with probability p and p_c, respectively in the supercritical and the critical cases, and closed otherwise.

Unfortunately we have not yet been able to find connection between the models below and invasion percolation as these models turned out to be rather complicated in their own right. As we pointed out in Chapter 1 it is not straightforward whether such processes exist, since the jump rates depend on the state of infinitely many edges. In the current chapter we define three different models and we show that they exist under certain conditions on the clock rates. We also propose to study some related open problems.

4.2 The models

The reader may find the definition of the incipient infinite cluster in the previous chapter, but here we briefly recall that definition. In the paper [33], Kesten defined the so called *incipient infinite cluster* (IIC) for two-dimensional percolation. The IIC is obtained by conditioning critical percolation, through a limiting process, to have an infinite cluster. Let $\mathbb{P}_p$ denote the probability measure governing Bernoulli bond percolation at density p, and let $B(n) = [-n, n]^2 \cap \mathbb{Z}^2$. Kesten showed that the following two limits exist and coincide for any cylinder event E:

$$\begin{aligned} \nu(E) &= \lim_{n\to\infty} \mathbb{P}_{p_c}(E|0 \longleftrightarrow B(n)^c) \\ &= \lim_{p\downarrow p_c} \mathbb{P}_p(E|0 \longleftrightarrow \infty). \end{aligned} \tag{4.2.1}$$

4.2.1 Description of the models

In this chapter, we will consider dynamical processes. We hope them to be well-defined Markov chains and to help us gain better insight to the IIC distribution. It is important to note again that the descriptions of the models below should only be treated as "informal definitions" because the existence of neither of these processes is clear at this point. Formal definition will be given in later sections where we prove the existence of these processes. In each of our models, an edge e is assigned a rate $\lambda_e > 0$ Poisson clock, independently of the clocks of the other edges.

Model A. For the supercritical model, we let $p > p_c$, and we take $\lambda_e \equiv 1$. The dynamics is started from a configuration that is distributed according to $\mathbb{P}_p(\cdot|0 \longleftrightarrow \infty)$. When the clock of an edge e rings, the following operation takes place:

1. If the edge is pivotal for $\{0 \longleftrightarrow \infty\}$, then nothing happens.
2. If the edge is not pivotal, we make it open with probability p or closed with $1 - p$.

Model B. In order to describe our second model, we introduce the following notation:

$$\pi_1(n) = \mathbb{P}_{p_c}(\exists \text{ an open path from } 0 \text{ to } B(n)^c);$$

$$\pi_3(n) = \mathbb{P}_{p_c}\begin{pmatrix}\exists \text{ two disjoint open paths and one closed path from} \\ \text{the origin to } B(n)^c\end{pmatrix}.$$

We take $p = p_c$, and λ_e satisfying

$$M_3 := \sum_{e \in \mathbb{E}^2} \lambda_e \pi_3(|e|) < \infty, \tag{4.2.2}$$

where the sum is over all edges of $\mathbb{Z}^2$. The process is started from the initial measure ν. The dynamics governing the process, namely the updating procedure of the state of an edge when its clock rings, is the same as that in Model A: edges pivotal for $\{0 \longleftrightarrow \infty\}$ remain unchanged while the state of non-pivotal edges are redrawn according to the probability p_c.

Although we construct the process under condition (4.2.2), it will be easier to first carry out the construction under the more restrictive condition

$$M_1 := \sum_{e \in \mathbb{E}^2} \lambda_e \pi_1(|e|) < \infty. \tag{4.2.3}$$

Model C. In the third model we consider a process where the cluster changes in a non-local fashion. We again take $p = p_c$, and λ_e satisfying (4.2.2). For a configuration η we write $\mathcal{C}_\eta(0)$ for the cluster of the origin in η. We may drop η from our notation when it is clear from the context which configuration is considered. We recall the definition of the outer edge boundary ΔG of an arbitrary subgraph G of $\mathbb{Z}^2$ from Section 3.1.1:

$$\Delta G = \{e = \langle x, y\rangle \in \mathbb{E}^2 \ : \ e \notin E(G), \text{ but } x \in G \text{ or } y \in G\}.$$

For an edge $e \in \mathcal{C}_\eta(0)$ that is pivotal for $\{0 \longleftrightarrow \infty\}$ in η we define

$$E_e[\eta] = \{e\} \cup \{f \in \mathcal{C}(0) \cup \Delta\mathcal{C}(0) : 0 \longleftrightarrow \infty \text{ in } (\eta \setminus \{e\}) \cup \{f\}\}.$$

So $E_e[\eta]$ is the set of edges f such that if we make e closed but f open, then the origin will remain connected to infinity.

Again, each edge e has a Poisson-clock attached to it with rate λ_e. When the clock rings, the following operation takes place:

1. We make the edge open with probability p_c or closed with $1 - p_c$. If it becomes open we do not do anything else.
2. If it becomes closed we check if it was pivotal for $\{0 \leftrightarrow \infty\}$. If not we do nothing else.
3. If it was pivotal, then consider the set of edges $E_e[\eta]$. We choose an edge f from $E_e[\eta]$ with probability

$$q(f) := \frac{\lambda_f}{\sum_{f' \in E_e[\eta]} \lambda_{f'}}, \tag{4.2.4}$$

and make it open.

4.2.2 Motivation behind the models

We have already mentioned our main goals with introducing these dynamical processes but since we give a total of three different models it is natural to mention why each particular process is of interest and why it is worth to study all of them instead of only one.

It is rather immediate why Model A is defined. It is the only process whose existence we are able to prove under the unrestrictive condition that $\lambda_e \equiv 1$ for all edge $e \in \mathbb{Z}^2$. However, our proof strongly relies on taking $\mathbb{P}_p(\cdot|0 \longleftrightarrow \infty)$ to be the initial measure and p being the probability used when updating. Therefore we cannot expect the IIC measure to be a stationary measure for Model A because we do not even know if the process, started with initial configuration distributed according to $\nu(\cdot)$, exists under such mild condition on the rates. Our aim with introducing this model is to later study the behaviour of the limiting process as $p \downarrow p_c$.

The main reason to consider Model B is that it does have the IIC measure as an invariant measure and we conjecture the process to converge to ν, see Conjecture 4.6.1 below. However, constructing the process in Model B comes at a price of stricter condition on the rates. While this is not really desirable we argue that condition (4.2.2) still allows for an interesting model. To justify this, we first need to point out that under (4.2.3) the model is not too interesting, since then with probability one in any finite time interval only finitely many edges of $\mathcal{C}_{\eta_0}(0)$ attempt to flip. This follows from $\mathbb{P}(e \in \mathcal{C}(0) \cup \Delta\mathcal{C}(0)) \asymp \pi_1(|e|)$, which can be shown using standard percolation arguments. Therefore, for every $t > 0$ there is a box of size $n < \infty$ such that $\mathcal{C}_{\eta_t}(0) \cap B(n)^c = \mathcal{C}_{\eta_0}(0) \cap B(n)^c$. However, under condition (4.2.2) in any finite time interval with probability one infinitely many edges of $\mathcal{C}_{\eta_0}(0)$ attempt to flip. In fact, one can also show that infinitely many edges of the so called backbone of $\mathcal{C}_{\eta_0}(0)$ want to update. We define the backbone $\mathcal{B}$ of an infinite connected subgraph $\mathcal{C} \subset \mathbb{Z}^2$ that contains the origin:

1. The vertex set of $\mathcal{B}$ consists of all the sites $v \in \mathcal{C}$ such that there exists two open paths starting from v that connect v to the origin and to infinity and these paths are disjoint apart from their common starting point.
2. An edge $e \in \mathbb{Z}^2$ is in the backbone if $e \in \mathcal{C}$ and it connects two vertices of the backbone.

In many applications the backbone is the most important part of the infinite cluster, see for instance [4]. To show that assuming (4.2.2), infinitely many edges of the backbone flip in any finite time interval, note that standard gluing argument from independent percolation implies that a vertex v that is at distance n from the origin is on the backbone of the IIC with probability proportional to $\pi_3(n)/\mathbb{P}_{p_c}(0 \longleftrightarrow \partial B(n))$.

Finally we also study Model C, because on the one hand, it may be more suitable to find connection with an appropriate dynamics for the IPC, while on the other it may converge faster to its stationary measure due to its non-locally changing backbone.

Note that finite volume versions of each model can be defined easily, and it is straightforward to verify that each process is reversible: in the case of Model A with respect to $\mathbb{P}_p(\cdot|0 \longleftrightarrow B(n)^c)$, and in the case of Models B, C, with respect to $\mathbb{P}_{p_c}(\cdot|0 \longleftrightarrow B(n)^c)$. In the infinite volume case, the jump rates depend non-locally on the configuration, hence it is not immediate how to construct the corresponding Markov process. We will carry out the constructions in the next section.

Finally, we introduce the following notation, which will be used throughout this chapter: for a configuration η and edges $e \neq f$, we let

$$\eta^e(g) = \begin{cases} 1 & \text{if } g = e; \\ \eta(g) & \text{if } g \neq e; \end{cases}$$

$$\eta_f(g) = \begin{cases} 0 & \text{if } g = f; \\ \eta(g) & \text{if } g \neq f; \end{cases}$$

$$\eta_f^e(g) = \begin{cases} 1 & \text{if } g = e; \\ 0 & \text{if } g = f; \\ \eta(g) & \text{if } g \neq e, f. \end{cases}$$

4.3 Construction of the models

4.3.1 The supercritical model

First, we construct the process described by Model A. We start with a formal definition.

Definition 4.3.1. *Let $\chi_e^A : \mathbb{R}_+ \to \{0,1\}, e \in \mathbb{E}^2$ be a collection of processes defined jointly on some probability space with joint distribution η_t^A and satisfying the following properties:*

(i) η_0^A *is distributed as* $\mathbb{P}_p(.|0 \leftrightarrow \infty)$.

(ii) Almost surely for all $e \in \mathbb{E}^2, t \to \chi_e^A(t)$ is right continuous with left limits (c.a.d.l.a.g.).

(iii) For each $e \in \mathbb{E}^2$ there exists a set of random times $\{T_i^{(e)}\}_{i\in\mathbb{N}}$ such that $\{T_{i+1}^{(e)} - T_i^{(e)}\}_{i\in\mathbb{N}}$ are independent, exponentially distributed with mean 1. Furthermore, the value of $\chi_e^A(t)$ is constant within each interval $T_{i+1}^{(e)} - T_i^{(e)}$.

(iv) If $\chi_e^A(T_i^{(e)-}) = 0$ then $\chi_e^A(T_i^{(e)}) = 1$. If $\chi_e^A(T_i^{(e)-}) = 1$ and e is pivotal for the event $\{0 \leftrightarrow \infty\}$ in the configuration $\eta^A_{T_i^{(e)-}}$ then $\chi_e^A(T_i^{(e)}) = 1$. Otherwise $\chi_e^A(T_i^{(e)}) = 0$.

As we pointed out earlier it is unclear whether there are processes satisfying all the conditions in Definition 4.3.1. In the following theorem we state that this is indeed the case.

Theorem 4.3.2. *There is a collection of stochastic processes* $\chi_e^A : \mathbb{R}_+ \to \{0,1\}, e \in \mathbb{E}^2$ *that satisfies all the requirements of Definition 4.3.1.*

Proof. It is sufficient to construct the joint distribution η_t^A for all $t \geq 0$. In fact, it is enough to construct η_t^A on a fixed time-interval $[0, t_0]$, $t_0 > 0$ and show that $\eta_{t_0}^A$ has again distribution $\mathbb{P}_p(\cdot | 0 \longleftrightarrow \infty)$.

Let $\nu_p(\cdot) := \mathbb{P}_p(\cdot | 0 \longleftrightarrow \infty)$. Due to the FKG-inequality for Bernoulli percolation (see Appendix), ν_p is stochastically larger than $\mathbb{P}_p$. Let $\mathcal{E}$ denote the set of edges that are open initially and do not attempt to become vacant some time between $[0, t_0]$. During $[0, t_0]$, there is probability $\delta := 1 - \exp(-(1-p)t_0)$ that a given edge e attempts to become vacant. Hence the distribution of $\mathcal{E}$ is stochastically larger than $\mathbb{P}_{p-\delta}$. We select t_0 small enough such that $p' := p - \delta$ satisfies $p' > p_c$. Hence, apriori, the distribution of the configuration of the process is bounded below stochastically by $\mathbb{P}_{p'}$ throughout $[0, t_0]$. Let $\mathbb{P}$ be a monotone coupling between ν_p, $\mathcal{E}$ and $\mathbb{P}_{p'}$. Let η_0 and ξ denote the ν_p- and $\mathbb{P}_{p'}$-marginals.

Just like in Chapter 3, we define $A(m,n)$ to be the event that there is an open circuit in the annulus $B(n) \setminus B(m)$. By estimates on supercritical percolation, using standard techniques such as the RSW theorem and FKG inequality, one can show that with probability one there exists an n such that the following event occurs in ξ:

$$A'(n) := \left\{ \begin{matrix} A(n,2n) \text{ and } A(2n,4n) \text{ occur and } \exists \text{ two} \\ \text{disjoint open paths from } B(n) \text{ to } \infty \end{matrix} \right\}.$$

We define the process $\{\eta_t\}_{0 \leq t \leq t_0}$ separately on $B(2n)$ and $B(2n)^c$.

On $B(2n)^c$ all jump attempts are accepted, that is, when an edge attempts to become vacant, it does become vacant. Due to the coupling with ξ, and the event $A'(n)$, for all $0 \leq t \leq t_0$, there are no pivotal edges for $0 \longleftrightarrow \infty$ in configuration η_t in $B(2n)^c$. Note that by construction, $\eta_t \geq \xi$ in $B(2n)^c$.

On $B(2n)$, there are finitely many Poisson events during $[0, t_0]$, that can be examined one-by-one. An attempt to become open is always accepted, and an attempt to become closed is suppressed if the edge is pivotal for $0 \longleftrightarrow B(4n)$. Due to the event $A'(n)$, this happens if and only if the edge is pivotal for $0 \longleftrightarrow \infty$.

Therefore, with probability 1, on $[0, t_0]$ the jump rate of each edge e depends on finitely many edges only. If $e \in B(2n)^c$ then it depends on e only. If $e \in B(2n)$, the jump rate depends on the configuration inside $B(4n)$.

To complete the construction, we need to show that the distribution of the process at time t_0 is again $\nu_p(\cdot)$, which would enable us to iterate the above construction for the time interval $[t_0, 2t_0]$ and so on. The proof of this claim is based on finite volume approximation and it is very similar to but somewhat simpler than the proof of Lemma 4.3.7 and Lemma 4.3.12 below. Since these two lemmas are proved in details we omit to give a proof here. □

4.3.2 Infinite cluster is unchanged for a positive time

Next, we construct the process described by Model B. We again begin with the formal definition.

Definition 4.3.3. *Let $\chi_e^B : \mathbb{R}_+ \to \{0,1\}, e \in \mathbb{E}^2$ be a collection of processes defined jointly on some probability space with joint distribution η_t^B and satisfying the following properties:*

(i) *η_0^B is distributed as the IIC measure ν..*

(ii) *Almost surely for all $e \in \mathbb{E}^2, t \to \chi_e^B(t)$ is right continuous with left limits (c.a.d.l.a.g.).*

(iii) *For each $e \in \mathbb{E}^2$ there exists a set of random times $\{T_i^{(e)}\}_{i\in\mathbb{N}}$ such that $\{T_{i+1}^{(e)} - T_i^{(e)}\}_{i\in\mathbb{N}}$ are independent, exponentially distributed with some mean $0 < \lambda_e < \infty$. Furthermore, the value of $\chi_e^B(t)$ is constant within each interval $T_{i+1}^{(e)} - T_i^{(e)}$.*

(iv) *If $\chi_e^B(T_i^{(e)-}) = 0$ then $\chi_e^B(T_i^{(e)}) = 1$. If $\chi_e^B(T_i^{(e)-}) = 1$ and e is pivotal for the event $\{0 \leftrightarrow \infty\}$ in the configuration $\eta^B_{T_i^{(e)-}}$ then $\chi_e^B(T_i^{(e)}) = 1$. Otherwise $\chi_e^B(T_i^{(e)}) = 0$.*

Theorem 4.3.4. *If the rates, λ_e, satisfy condition* (4.2.3)*, that is if*

$$\sum_{e\in\mathbb{E}^2} \lambda_e \pi_1(|e|) < \infty$$

then there exists a collection of stochastic processes satisfying the requirements in Definition 4.3.3.

Proof. Condition (4.2.3) ensures that the cluster $\mathcal{C}(0)$ will not change up to a positive time. Given an initial configuration ξ_0 distributed according to ν, we construct the process on a random time interval $[0, T_1]$. The time T_1 will be the first time when the cluster $\mathcal{C}(0)$ or one of its boundary edges wants to flip. Up to this time, the configuration in the complement of $\mathcal{C}(0)$ evolves with edges flipping according to independent two-state Markov chains. At time T_1, the edge in $\mathcal{C}(0) \cup \partial_e\mathcal{C}(0)$ flips, unless it was pivotal for $0 \longleftrightarrow \infty$. We now give a rigorous construction.

Let

$$\mathcal{E}_1 = \mathcal{C}(0;\xi_0) \cup \Delta\mathcal{C}(0;\xi_0),$$

with $\Delta\mathcal{C}(0;\xi_0)$ denoting the outer edge boundary of $\mathcal{C}(0;\xi)$ as defined in Chapter 3. Let

$$L_1 = \sum_{e\in\mathcal{E}_1} \lambda_e$$
$$p_1(e) = L_1^{-1}\lambda_e, \quad \text{for } e \in \mathcal{E}_1.$$

By (4.2.3) we have

$$\mathbb{E}_\nu(L_1) = \sum_{e\in\mathbb{E}^2} \lambda_e \nu(e\in\mathcal{E}_1) \asymp \sum_{e\in\mathbb{E}^2} \lambda_e \pi_1(|e|) \le M_1,$$

where we used that $\nu(e\in\mathcal{E}_1) \asymp \pi_1(|e|)$, which is a straightforward consequence of standard RSW arguments for Benoulli bond percolation. In particular, L_1 is finite with probability one, and $p_1(e)$ is well-defined.

We consider the following objects, all mutually independent, given ξ_0:

(i) Let $\{\zeta_{1,t}\}_{t\ge 0}$ be a collection of independent two-state Markov chains indexed by the edges in $\mathbb{E}^2\setminus\mathcal{E}_1$, where $\zeta_{1,0} = \xi_0|_{\mathbb{E}^2\setminus\mathcal{E}_1}$. The e-component of ζ_1 jumps from closed to open with rate $p_c\lambda_e$, and from open to closed with rate $(1-p_c)\lambda_e$.

(ii) Let T_1 be a random variable whose conditional distribution given ξ_0 is exponential with parameter L_1.

(iii) Let f_1 be a random edge of $\mathcal{E}_1$ selected according to the distribution p_1.

(iv) Let X_1 be a random variable with $\mathbb{P}(X_1=1)=p_c=1-\mathbb{P}(X_1=0)$.

We now define ζ_1' as the configuration in $\mathcal{E}_1$ with

$$\zeta_1' = \begin{cases} (\xi_0|_{\mathcal{E}_1})^{f_1} & \text{if } X_1 = 1; \\ (\xi_0|_{\mathcal{E}_1})_{f_1} & \text{if } X_1 = 0 \text{ and } f_1 \text{ is not pivotal for } 0 \longleftrightarrow \infty; \\ \xi_0|_{\mathcal{E}_1} & \text{otherwise.} \end{cases} \tag{4.3.1}$$

We finally set $\xi_1 = \zeta_{1,T_1} \vee \zeta_1'$, and

$$\eta_t^B = \begin{cases} \zeta_{1,t} \vee \xi_0 & \text{if } 0 \le t < T_1; \\ \xi_1 & \text{if } t = T_1. \end{cases}$$

Before we can prove that ξ_1 has distribution ν, we need some classical results concerning the IIC measure. Since the proofs of these results are relatively short, we include them here for the sake of completeness.

Lemma 4.3.5. *Let $\mathcal{D}$ be a deterministic circuit in the annulus $A(m,n)$ and condition the IIC measure on the event that $\mathcal{D}$ is the outermost open circuit in $B(n)\setminus B(m)$. Then the configuration in* int$\mathcal{D}$ *has the distribution of critical percolation conditioned on the event $\{0 \longleftrightarrow \mathcal{D}\}$.*

Proof. Let $F_\mathcal{D}$ denote the event that $\mathcal{D}$ is the outermost open circuit in $B(n)\setminus B(m)$. Let $n' > n$, and let η be a configuration in int$\mathcal{D}$ such that the origin is connected to $\mathcal{D}$ in η. Then

$$\begin{aligned}\mathbb{P}_{p_c}(\eta,\, F_\mathcal{D},\, 0 \longleftrightarrow B(n')^c) &= \mathbb{P}_{p_c}(\eta,\, F_\mathcal{D},\, \mathcal{D} \longleftrightarrow B(n')^c) \\ &= \mathbb{P}_{p_c}(\eta)\mathbb{P}_{p_c}(F_\mathcal{D},\, \mathcal{D} \longleftrightarrow B(n')^c).\end{aligned}$$

Dividing by $\mathbb{P}_{p_c}(F_{\mathcal{D}}, 0 \longleftrightarrow B(n')^c)$ and letting $n' \to \infty$, the claim follows by (4.2.1). ☐

Lemma 4.3.6. *The IIC measure has the property that conditioned on the cluster of the origin $\mathcal{C}(0)$, the configuration in $\mathbb{E}^2 \setminus (\mathcal{C}(0) \cup \Delta\mathcal{C}(0))$ is i.i.d. with edge density p_c.*

Proof. Let again $\mathcal{D}$ be a deterministic circuit in the annulus $A(m, n)$. Using Lemma 4.3.5, it can be verified directly that conditioned on $F_{\mathcal{D}}$ and $\mathcal{C}(0) \cap \text{int}\mathcal{D}$, the statement holds for the configuration in $\text{int}\mathcal{D}$. Since for any m we have

$$\lim_{n \to \infty} \nu(\cup_{\mathcal{D}} F_{\mathcal{D}}) = 1,$$

it follows that all finite-dimensional marginals have the claimed distribution. ☐

Now we are ready to prove that at time T_1 the distribution of the process is the IIC measure.

Lemma 4.3.7. *The configuration $\eta_{T_1} = \xi_1$ has distribution ν.*

Proof. By Lemma 4.3.6 we have that given $\mathcal{E}_1$, the configuration $\zeta_{1,0}$ is i.i.d. with edge density p_c. The two-state Markov chains are hence stationary, and it follows that ζ_{1,T_1} is again i.i.d. with edge density p_c. To complete the proof, we need to show that the operation made at time T_1 leaves ν invariant. It is sufficient to show this conditioned on f_1, where f_1 is the edge that is selected to flip at time T_1. That is, we need to show that for any fixed $f_1 \in \mathcal{E}_1$, the operation in (4.3.1) leaves ν invariant.

We use a finite volume approximation to prove this. Let m be large, so that $f_1 \in B(m)$. Let $n > m$ and decompose the event $A(m, n)$ according to the outermost circuit in the annulus $B(n) \setminus B(m)$:

$$\begin{aligned} A(m,n) &= \bigcup_{\mathcal{D}} F_{\mathcal{D}} \\ &:= \bigcup_{\mathcal{D}} \{\mathcal{D} \text{ is the outermost open circuit in } B(n) \setminus B(m)\}. \end{aligned}$$

Lemma 4.3.5 implies the property that conditioned on $F_{\mathcal{D}}$, the probability of seeing a configuration η in $\text{int}(\mathcal{D})$ is proportional to the weight $w(\eta) = p_c^N (1 - p_c)^M$, where N and M are the number of open and closed edges in η, respectively. We denote the normalizing constant by $Z_{\mathcal{D}}$.

Since the operation (4.3.1) leaves the edges in $B(n) \setminus B(m)$ unchanged, the event that $A(m, n)$ (or $F_{\mathcal{D}}$) occurs on ξ_0 is the same as the event that $A(m, n)$ (or $F_{\mathcal{D}}$) occurs on ξ_1.

Fix $\mathcal{D}$, and consider a configuration η in $\text{int}(\mathcal{D})$ such that f_1 is pivotal for $0 \longleftrightarrow \mathcal{D}$ in η. Note that if $F_{\mathcal{D}}$ occurs in ξ_0 and ξ_0 agrees with η in $\text{int}(\mathcal{D})$, then f_1 is pivotal for $0 \longleftrightarrow \infty$ in ξ_0. Therefore we have $\{F_{\mathcal{D}}$ in ξ_1, $\xi_1|_{\text{int}(\mathcal{D})} =$

$\eta\} = \{F_{\mathcal{D}} \text{ in } \xi_0,\, \xi_0\big|_{\mathrm{int}(\mathcal{D})} = \eta\}$. In particular, these two events have the same probability.

Consider now a configuration η in $\mathrm{int}(\mathcal{D})$ such that $\eta(f_1) = 0$. Then f_1 is not pivotal for $0 \longleftrightarrow \mathcal{D}$, and hence if $F_{\mathcal{D}}$ occurs in ξ_0 and if ξ_0 agrees with η in $\mathrm{int}(\mathcal{D}) \setminus \{f_1\}$, then f_1 will not be pivotal for $0 \longleftrightarrow \infty$ in ξ_0. Hence we have

$$
\begin{aligned}
&\mathbb{P}(F_{\mathcal{D}} \text{ in } \xi_1,\, \xi_1\big|_{\mathrm{int}(\mathcal{D})} = \eta) \\
&\quad = \mathbb{P}(F_{\mathcal{D}} \text{ in } \xi_0,\, X_1 = 0,\, \xi_0\big|_{\mathrm{int}(\mathcal{D})} = \eta_{f_1}) + \mathbb{P}(F_{\mathcal{D}} \text{ in } \xi_0,\, X_1 = 0,\, \xi_0\big|_{\mathrm{int}(\mathcal{D})} = \eta^{f_1}) \\
&\quad = \mathbb{P}(F_{\mathcal{D}} \text{ in } \xi_0)\left[(1-p_c)\frac{1}{Z_{\mathcal{D}}} w(\eta_{f_1}) + (1-p_c)\frac{1}{Z_{\mathcal{D}}} w(\eta^{f_1})\right] \\
&\quad = \mathbb{P}(F_{\mathcal{D}} \text{ in } \xi_0)\frac{1}{Z_{\mathcal{D}}} w(\eta) \\
&\quad = \mathbb{P}(F_{\mathcal{D}} \text{ in } \xi_0,\, \xi_0\big|_{\mathrm{int}(\mathcal{D})} = \eta).
\end{aligned}
$$

Similar considerations apply to a configuration η such that $\eta(f_1) = 1$.

We have proved invariance conditioned on the event $A(m,n)$. Since $\cup_n A(m,n)$ has ν-probability 1, and the events $A(m,n)$ are themselves invariant, this proves invariance of ν. □

We now analogously construct, starting from the configuration ξ_1, the random data $\mathcal{E}_2$, L_2, T_2, $\{\zeta_{2,t}\}_{T_1 \le t \le T_1 + T_2}$, ζ_2', and ζ_2'', and from these a new configuration ξ_2, and so on.

Lemma 4.3.8. *We have $\sum_{i=1}^n T_i \to \infty$ almost surely.*

Proof. By construction, the sequence $L_1, L_2, \dots$ is stationary. We have

$$
\sum_{i=1}^n \frac{1}{L_i} \ge \frac{n}{\frac{1}{n}\sum_{i=1}^n L_i}.
$$

Since $\mathbb{E}(L_1) \le M < \infty$, the ergodic theorem implies that the denominator on the right hand side converges to an a.s. finite limit. This implies that $\sum_{i=1}^n L_i^{-1}$ goes to ∞ a.s. Since given $L_1, L_2, \dots$, the variables $T_1, T_2, \dots$ are conditionally independent exponentials with rates $L_1, L_2, \dots$, this implies the lemma. □

Since at time T_1 the configuration is distributed again according to ν, and $T_n \to \infty$ almost surely, this completes the construction of the process $\{\eta_t\}_{t \ge 0}$. □

4.3.3 Open circuits remain for a positive time

In the previous Section we proved the existence of a process described as Model B under the rather restrictive condition (4.2.3). As we pointed out earlier, this condition implies that only finitely many edges of the infinite cluster want to flip in any finite length time interval. Since we would like to ease this restriction we study the existence of the process under condition (4.2.2).

Theorem 4.3.9. *If the rates, λ_e, satisfy condition* (4.2.2)*, that is if*

$$\sum_{e\in\mathbb{E}^2} \lambda_e \pi_3(|e|) < \infty$$

then there exists a collection of stochastic processes satisfying the requirements in Definition 4.3.3.

Proof. The main idea is the following: Since the process is started from the measure ν, initially there are infinitely many open circuits surrounding the origin. This can be seen using the FKG inequality, which implies that ν stochastically dominates $\mathbb{P}_{p_c}$, and the RSW theorem, which implies the existence of the circuits in critical Bernoulli percolation. We use condition to ensure that infinitely many open circuits remain present for a positive time. Finally, we use an argument very similar to that in the previous section to show how the existence of these circuits help defining our process.

To carry out this plan we first consider an appropriate subset of the cluster $\mathcal{C}(0)$.

Let ξ_0 be the initial configuration distributed according to ν. Abbreviate $A_k = B(2^k) \setminus B(2^{k-1})$, and $A'_k = B(\frac{7}{8}2^k) \setminus B(\frac{5}{8}2^k)$ where $k \geq 4$. For $e \in A'_k$, we define the event

$$D_3(e) := \begin{Bmatrix} \exists \text{ open circuit in } A_k \text{ containing } e \text{ and surrounding} \\ B(2^{k-1}), \text{ and } \exists \text{ closed dual path from } e \text{ to } B(e, 2^{k-3})^c \end{Bmatrix},$$

where $B(e, 2^{k-3})$ is the box of size 2^{k-3} around, i.e. $B(e, 2^{k-3}) = e + B(2^{k-3})$.

$$\mathcal{E}_1 := \cup_{k=2}^{\infty} \{e \in A_k : D_3(e) \text{ occurs}\}.$$

Lemma 4.3.10. *We have*

$$\nu(D_3(e)) \asymp \pi_3(|e|).$$

Proof. We note first that $\mathbb{P}_{p_c}(D_3(e)) \leq \pi_3(2^{k-3}) \leq C_1\pi_3(|e|)$. Writing $D_1(n)$ for the event $\{0 \longleftrightarrow B(n)^c\}$, for $n > 2^{k+1}$ we have, by the RSW theorem

$$\begin{aligned} \mathbb{P}_{p_c}(D_3(e),\, D_1(n)) &\leq \mathbb{P}_{p_c}(D_1(2^{k-3}),\, D_3(e),\, B(2^k) \longleftrightarrow B(n)^c) \\ &\leq \mathbb{P}_{p_c}(D_1(2^{k-3}))\mathbb{P}_{p_c}(D_3(e))\mathbb{P}_{p_c}(B(2^k) \longleftrightarrow B(n)^c) \\ &\leq C_2\pi_3(|e|)\mathbb{P}_{p_c}(D_1(n)). \end{aligned}$$

Dividing by $\mathbb{P}_{p_c}(D_1(n))$ yields that $\nu(D_3(e)) \leq C_4\pi_3(|e|)$.

To prove the inequality in the other direction, we consider the event that there exists two open paths and one closed dual path from e to $B(e, 2^{k-3})^c$.

Using a standard arm extension technique due to Kesten [34] the two open arms can be extended to an open circuit. The claim follows from a standard RSW argument. □

For the rigorous construction we again let

$$L_1 = \sum_{e \in \mathcal{E}_1} \lambda_e$$
$$p_1(e) = L_1^{-1}\lambda_e, \qquad e \in \mathcal{E}_1.$$

By Lemma 4.3.10, we have

$$\mathbb{E}_\nu(L_1) = \sum_{e \in \mathbb{E}^2} \lambda_e \nu(e \in \mathcal{E}_1) \le C \sum_{e \in \mathbb{E}^2} \lambda_e \pi_3(|e|) \le M_3.$$

Hence L_1 is finite with probability 1, and $p_1(e)$ is well-defined.

Consider the first time T_1 when an edge in $\mathcal{E}_1$ wants to flip. Conditioned on ξ_0, T_1 is exponential with rate L_1. Let f_1 be the edge that wants to flip at time T_1. We first construct the process on $[0, T_1)$.

Let $A_k'' := B(2^k) \backslash B(\frac{7}{8}2^k)$. We define the dual lattice of $\mathbb{Z}^2$ as $(1/2, 1/2)+\mathbb{Z}^2$, that is, a translated version of the original lattice, where the vector of the shift is $(1/2, 1/2)$. Each edge of $\mathbb{Z}^2$ is crossed by exactly one edge of the dual lattice, which enables us to find a one-to-one correspondence between the original and the dual lattice: an edge of the dual is open if and only if the corresponding edge in the original lattice is open. A simple geometric argument shows that there is an open circuit in $B(n) \setminus B(m)$ in the original lattice if and only if there is no closed dual crossing of the same annulus. We call a dual circuit an *almost* closed circuit, if all of its edges are closed, except for one. Consider the event:

$$E_k := \left\{ \begin{matrix} \exists \text{ open circuit in } A_k',\ \exists \text{ almost closed dual circuit in} \\ A_k'',\ \exists \text{ closed dual crossing of } A_k'' \end{matrix} \right\}$$

Lemma 4.3.11. *(i) We have $\nu(E_k$ holds for infinitely many $k) = 1$. (ii) Suppose that E_k occurs in ξ_0 for some $k > 0$. Let $\mathcal{D}$ be the outermost open circuit in A_k. Then $\mathcal{D} \subset \mathcal{E}_1$, and hence there is no jump attempt for any edge of $\mathcal{D}$ in $[0, T_1)$.*

Proof. For a fixed $k > 0$ we have that $\nu(E_k) \ge C_1 > 0$, which follows from standard RSW argument for critical Bernoulli percolation. Since E_k is determined by the state of edges within $A_k = B(2^k) \setminus B(2^{k-1})$, this proves *(i)*. *(ii)* follows from the definition of $\mathcal{E}_1$. □

Fix a sequence $k_1 < k_2 < \dots$ such that E_{k_i} occurs in ξ_0, for $i = 1, 2, \dots$. Let $\mathcal{D}(k_i)$ be the outermost open circuit in A_{k_i}'. We condition on E_{k_i} and $\mathcal{D}(k_i)$. There are finitely many jump attempts in $[0, T_i)$ in $\text{int}\mathcal{D}(k_i)$. We define $\{\eta_t^{(k_i)}\}_{0 \le t < T_i}$ as the following process on edges inside $\mathcal{D}(k_i)$. The initial configuration is $\eta_0^{(k_i)} = \xi_0\big|_{\mathrm{int}\mathcal{D}(k_i)}$. Jump attempts to become open are all accepted, and jump attempts to become closed are accepted if the edge is not pivotal for $0 \longleftrightarrow \mathcal{D}(k_i)$. We couple the processes for different i using the same Poisson jump processes. Then it is straightforward to verify that the definitions are consistent in that $\eta_t^{(k_j)}\big|_{\mathrm{int}\mathcal{D}(k_i)} = \eta_t^{(k_i)}$ for $j > i$. Hence there is a unique process $\{\eta_t\}_{0\le t<T_i}$ extending this family. The process $\{\eta_t\}_{0\le t<T_i}$ is also independent of the choice the sequence $k_1 < k_2 < \dots$.

Note that in η_{T_1-}, all the open circuits $\mathcal{D}(k_i)$ are present. Let i be such that $f_1 \in \mathrm{int}\mathcal{D}(k_i)$. We define

$$\xi_1(e) = \begin{cases} \eta_{T_1-}(e) & \text{if } e \neq f_1; \\ 1 & \text{if } e = f_1,\ X_1 = 1; \\ 0 & \text{if } e = f_1,\ X_1 = 0,\ f_1 \text{ not pivotal for } 0 \longleftrightarrow \mathcal{D}(k_i); \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 4.3.12. *ξ_1 has distribution ν.*

Proof. The proof is similar to the proof of Lemma 4.3.7. Let $m < n$, $f_1 \in B(m)$, and condition on the event $\cup_{k:2m\le 2^k\le n} E_k$. Let k^* be the largest $k, 2m \le 2^k \le 2n$, for which E_k occurs, and condition on the outermost open circuit $\mathcal{D}(k^*)$. Observe that the conditional distribution of ξ_0 in $\mathrm{int}\mathcal{D}(k^*)$ is critical percolation conditioned on the event $\{0 \longleftrightarrow \mathcal{D}(k^*)\}$. The dynamics of $\{\eta_t^{(k^*)}\}$ is reversible with respect to this measure, hence $\eta_{T_1}^{(k^*)}$ again has this distribution. We see as in Lemma 4.3.7, that the potential flip of f_1 leaves the measure invariant. Since m, n can be chosen arbitrarily large, the claim follows. □

The proof that $\sum_{i=1}^n T_i \to \infty$ a.s. can be carried out as before. □

4.4 The critical model with non-local changes

In this section we construct the process in Model C. Since it is essentially the same as that of Model B we will only treat the construction under assuming (4.2.3). The construction below can also be carried out in the case of the weaker assumption (4.2.3). Before the formal definition, recall the definition of $E_e(\eta)$ and $q(f)$ from Section 4.2.1.

Definition 4.4.1. *Let $\chi_e^C : \mathbb{R}_+ \to \{0,1\}, e \in \mathbb{E}^2$ be a collection of processes defined jointly on some probability space with joint distribution η_t^C and satisfying the following properties:*

(i) η_0^C is distributed as the IIC measure ν..

(ii) Almost surely for all $e \in \mathbb{E}^2, t \to \chi_e^C(t)$ is right continuous with left limits (c.a.d.l.a.g.).

(iii) *For each* $e \in \mathbb{E}^2$ *there exists a set of random times* $\{T_i^{(e)}\}_{i\in\mathbb{N}}$ *such that* $\{T_{i+1}^{(e)} - T_i^{(e)}\}_{i\in\mathbb{N}}$ *are independent, exponentially distributed with some mean* $0 < \lambda_e < \infty$. *Furthermore, the value of* $\chi_e^C(t)$ *is constant within each interval* $T_{i+1}^{(e)} - T_i^{(e)}$.

(iv) *If* $\chi_e^C(T_i^{(e)-}) = 0$ *then* $\chi_e^C(T_i^{(e)}) = 1$. *If* $\chi_e^B(T_i^{(e)-}) = 1$ *and* e *is not pivotal for the event* $\{0 \leftrightarrow \infty\}$ *in the configuration* $\eta^C_{T_i^{(e)-}}$ *then* $\chi_e^C(T_i^{(e)}) = 0$.

(v) *If* $\chi_e^B(T_i^{(e)-}) = 1$ *and* e *is pivotal for the event* $\{0 \leftrightarrow \infty\}$ *in the configuration* $\eta^C_{T_i^{(e)-}}$ *then* $\chi_e^C(T_i^{(e)}) = 0$ *and an edge* $f \in E_e[\eta^C_{T_i^{(e)-}}]$ *is selected with probability proportional to* $q(f)$ *and* $\chi_f^C(T_i^{(e)}) = 1$.

Theorem 4.4.2. *If the rates,* λ_e, *satisfy condition* (4.2.3), *that is if*

$$\sum_{e\in\mathbb{E}^2} \lambda_e \pi_1(|e|) < \infty$$

then there exists a collection of stochastic processes satisfying the requirements in Definition 4.4.1.

Proof. The construction is very similar to Model B. We define L_1, T_1, $\zeta_{1,t}$, f_1, X_1 and $\mathcal{E}_1$ as in Section 4.3.2. If f_1 is pivotal for $\{0 \longleftrightarrow \infty\}$, we set f_1^{new} to be an edge chosen according to the weight (4.2.4). For this, note that the denominator in (4.2.4) is bounded by L_1, hence the weights are well defined.

We now define ζ_1' as the configuration in $\mathcal{E}_1$ with

$$\zeta_1' = \begin{cases} (\xi_0|_{\mathcal{E}_1})^{f_1} & \text{if } X_1 = 1; \\ (\xi_0|_{\mathcal{E}_1})_{f_1} & \text{if } X_1 = 0 \text{ and } f_1 \text{ is not pivotal for } 0 \longleftrightarrow \infty; \\ (\xi_0|_{\mathcal{E}_1})_{f_1}^{f_1^{\text{new}}} & \text{if } X_1 = 0 \text{ and } f_1 \text{ is pivotal for } 0 \longleftrightarrow \infty. \end{cases} \tag{4.4.1}$$

We finally set $\xi_1 = \zeta_{1,T_1} \vee \zeta_1'$, and

$$\eta_t = \begin{cases} \zeta_{1,t} \vee \xi_0 & \text{if } 0 \le t < T_1; \\ \xi_1 & \text{if } t = T_1. \end{cases}$$

The construction can be carried out as for Model B, once we prove the following lemma.

Lemma 4.4.3. *The configuration* ξ_1 *has distribution* ν.

Proof. Again, it is sufficient to show that for fixed f_1, the operation in (4.4.1) leaves ν invariant, if f_1^{new} is chosen according to the weights (4.2.4). This can be done similarly to the proof of Lemma 4.3.7. For this, note that if $F_{\mathcal{D}}$ occurs, then f^{new} necessarily lies in $\text{int}(\mathcal{D})$. □

Lemma 4.4.3 concludes our construction on the random time interval $[0, T_1]$. The proof that $\sum_{i=1}^n T_i \to \infty$ a.s. can be carried out as before. □

4.5 Open problems

4.5.1 The lowest-crossing process

A natural question is whether the condition (4.2.2) can be relaxed, and the processes in Models B and C be defined for $\lambda_e \equiv 1$. One may try to do this directly, or study the limit of Model A as $p \downarrow p_c$.

A possible approach to a direct construction is to consider a thick annulus $B(n) \setminus B(m)$, such that initially there is an open circuit in this annulus. If we can show that an open circuit remains in the annulus for a positive time t_0, then inside $B(m)$ we can define our process during $[0, t_0]$. An initially existing open circuit may at some time be broken, by a bond becoming vacant. At this time, we can replace our circuit by a smaller one, as long as it still lies in the annulus.

The above motivates the following question. Instead of the annulus, for simplicity, we consider a vertical half strip of width n: $\mathcal{S}_n = [0, n] \times [0, \infty)$. Inside the strip, edges perform independent flips: $0 \to 1$ at rate p_c, $1 \to 0$ at rate $1 - p_c$. Let $\mathcal{D}_0$ be the lowest crossing in the initial configuration, and let $(\mathcal{D}_0)_-$ denote the set of edges lying below $\mathcal{D}_0$. At some positive time t, one of the edges of $\mathcal{D}_0$ will flip to vacant. At this time, we replace $\mathcal{D}_0$ by the lowest crossing in $\mathcal{S}_n \setminus (\mathcal{D}_0)_-$. We analogously change the crossing every time an edge of the current crossing becomes vacant. This defines a *lowest crossing process* $\{\mathcal{D}_t^n\}_{t \geq 0}$. We let L_t^n denote the vertical coordinate of leftmost point of $\mathcal{D}_t^n$.

Question. At what speed does the crossing move upwards, as a function of n? That is, what is the behaviour, as $n \to \infty$ of the speed

$$v(n) = \lim_{t \to \infty} \frac{L_t^n}{t}?$$

4.6 A convergence conjecture

We end with discussing some ideas regarding ergodicity of the constructed processes. We are interested in showing that the configuration in a fixed box $B(m)$ will be close to its distribution under ν, after sufficiently long time, depending on the rates of the process.

Consider for simplicity the process that only makes jumps in a large box $B(n)$, where $n > m$. Then the configuration ξ outside this box acts as a boundary condition for this process, which will approach its stationary distribution ν_ξ^n, which is the IIC measure conditioned on coinciding with ξ outside $B(n)$. We assume here that ξ is such that $\partial B(n) \longleftrightarrow \infty$ on ξ. Let $\mu_{\xi,t}^n$ denote the distribution of the process at time t, for some initial distribution μ_0^n. By the proof of Kesten's theorem, ν_ξ^n is close to ν on events in $\mathcal{F}_{B(m)}$, if n is large enough, uniformly in ξ. We expect that $\mu_{\xi,t}^n$ will reach this closeness at a rate that is uniform in ξ and μ_0^n. This suggests the following conjecture.

Conjecture 4.6.1. *Given rates* $\{\lambda_e\}$ *such that the infinite volume process exists, and given any* $\delta > 0$ *and* $m \geq 1$*, there exists* $n_0 = n_0(m, \delta, \{\lambda_e\})$ *and* $t_0 = t_0(m, \delta, \{\lambda_e\})$*, such that for all* $n \geq n_0$*, all* $t \geq t_0$ *and all initial measures* μ_0^n *we have*

$$\left|\mu_{\xi,t}^n(A) - \nu(A)\right| < \delta,$$

uniformly in $\xi \in \Omega_{B(n)^c}$ *and* $A \in \mathcal{F}_{B(m)}$.

Chapter 5

Appendix

In this appendix, we review a few results in ordinary percolation theory that played an important role throughout the research this thesis is concerned with. Although some of these results are valid also in higher dimensions we restrict the discussion to $\mathbb{Z}^2$. The first theorem to be mentioned here is due to Russo ([46]) and Seymour and Welsh ([48]). See also Section 11 of [26] for a detailed treatment of this theorem and some of its applications. The theorem concerns itself with the existence of open crossings of certain rectangles. Let $R(n, m) = [0, n] \times [0, m]$ and denote the event that there is a horizontal open crossing in $R(n, m)$ by $H(n, m)$.

Theorem 5.0.2. [RSW] *For each $\lambda > 0$ there exists a function f_λ satisfying $f_\lambda(x) \to 1$ as $x \to 1$ and $f_\lambda(x) > 0$ if $x > 0$ such that for all p and n*

$$\mathbb{P}_p(H(n, n)) > x \implies \mathbb{P}_p(H(\lambda n, n)) > f_\lambda(x).$$

Remark 5.0.3. *The translation invariance of the Bernoulli bond percolation model implies that if $\mathbb{P}_p(H(n, n)) > x$ then the probability of having a horizontal open crossing in an arbitrary λn by n rectangle is greater than $f_\lambda(x)$.*

The second result in this appendix is an analogue of Lemmma 2.2.1 for Bernoulli bond percolation. It was first proved by Harris in [29]. Later Fortuin, Kasteleyn and Ginibre ([24]) proved it in a more general context than stated here and therefore it became known as the *FKG*-inequality. Recall that an event A is called increasing if its indicator function I_A satisfies $I_A(\omega) \leq I_A(\omega')$ for any two configurations of edges with $\omega \leq \omega'$ coordinatewise.

Theorem 5.0.4. [FKG-inequality] *Consider Bernoulli bond percolation with arbitrary parameter $p \in [0, 1]$. Let A and B be two increasing events. Then*

$$\mathbb{P}_p(A \cap B) \geq \mathbb{P}_p(A)\mathbb{P}_p(B).$$

Finally, we conclude this Appendix with a consequence of Theorem 5.0.2 and 5.0.4. A simple symmetry argument shows that $\mathbb{P}_{1/2}(H(n,n)) \geq 1/2$ and hence by monotonicity this also holds for all $p \geq 1/2$. Therefore, for all $p \geq 1/2$ we can conclude from the RSW theorem that $\mathbb{P}_p(H(3n,n))$ is bounded away from 0 in n. Furthermore, note that using crossings of $3n$ by n rectangles we can construct a circuit in the annulus $B(3n) \setminus B(n)$ as it is pictured in Figure 5.1.

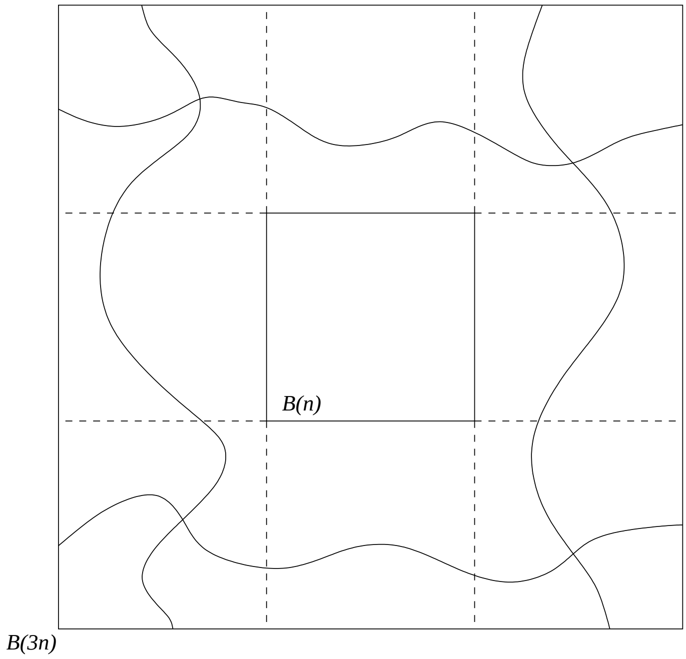

Figure 5.1: A circuit in $B(3n) \setminus B(n)$ can be constructed using crossings of appropriate rectangles.

It is easy to see that the event $H(3n,n)$ is increasing and hence we can apply the FKG inequality to show that the probability of having an open circuit in the annulus $B(3n) \setminus B(n)$ is again bounded away from 0. Since existence of crossings in disjoint annuli are independent a Borel-Cantelli argument implies the following Corollary, which we applied several times in this thesis.

Corollary 5.0.5. *Consider Bernoulli bond percolation on* $\mathbb{Z}^2$ *with parameter* $p \geq 1/2$. *Then with probability one the origin is surrounded by infinitely many open circuits.*

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Samenvatting

Zelf-destructieve percolatie, invasie percolatie en gerelateerde modellen

In dit proefschrift bestuderen we drie verschillende stochastische modellen: het zelf-destructieve percolatie model, het invasie percolatie model en een stochastische dynamica voor het emergent oneindige cluster model. Deze drie modellen zijn gedefinieerd op $\mathbb{Z}^2$. Dit is een graaf in het platte vlak met als puntenverzameling de punten in het platte vlak met gehele coördinaten en twee punten zijn verbonden in de graaf als hun Euclidische afstand gelijk is aan 1. Elk van de drie modellen geeft een bepaalde tijdsontwikkeling van deelgrafen van $\mathbb{Z}^2$.

Het eerste model, het zelf-destructieve percolatie model, is voortgekomen uit de studie van bepaalde bosbrandmodellen. Stel dat ieder punt van $\mathbb{Z}^2$ een plaats is waar een boom kan groeien. Beschouw een boom die gegroeid is op het punt v. Het cluster behorende bij deze boom is de maximale verzameling van bomen die bereikt kunnen worden vanuit v via een wandeling langs andere bomen waarbij de afstand tussen twee opeenvolgende bomen steeds precies 1 is. Een boom is in een oneindig cluster als het corresponderende cluster oneindig veel bomen bevat. We beginnen in de situatie waarin geen enkele boom aanwezig is. Vervolgens groeit op elk punt in $\mathbb{Z}^2$ een boom met kans $p \in [0,1]$, onafhankelijk van het groeien van bomen op andere plaatsen. Nadat op bovenstaande wijze het bos gecreeerd is, wordt een deel van het bos vernietigd, bijvoorbeeld door een bosbrand: als er oneindige clusters zijn dan worden alle bomen in deze clusters weer lege plaatsen in het bos. De eindige clusters worden behouden. Na de bosbrand groeien er weer bomen op elke lege plaats en de kans dat er na een bepaalde tijd een boom op een dergelijke plaats groeit is $\delta \in [0,1]$, wederom onafhankelijk van de andere locaties.

Een voor de hand liggende vraag is wat de kans is dat er een oneindig cluster bestaat in de uiteindelijke configuratie voor een gegeven paar (p, δ). We noteren deze kans met $\Theta(p, \delta)$. In Hoofdstuk 2 bestuderen we de functie $\Theta(p, \delta)$. Computer simulaties suggereren dat er een speciaal lijnsegment $\{\hat{p}\} \times [0, \hat{\delta}]$ bestaat zodanig dat $\Theta(.,.)$ niet continu is op $(\hat{p}, \hat{\delta})$. Helaas hebben we geen wiskundig bewijs hiervoor. Het belangrijkste resultaat van Hoofdstuk 2 is dat $\Theta(.,.)$ buiten het eerder genoemde lijnsegment wel continu is.

In het tweede model creëren we een stochastische deelgraaf van $\mathbb{Z}^2$ via een groeiproces dat op stochatische wijze verloopt. Iedere kant van $\mathbb{Z}^2$ krijgt een getal toegewezen volgens een bepaalde kansdichtheid. In de eerste stap behoort alleen een speciaal punt van $\mathbb{Z}^2$, genaamd de oorsprong, tot de deelgraaf. In elke volgende stap wordt een nieuwe kant en een nieuw punt toegevoegd volgens het volgende principe: we bekijken de getallen die aan de kanten op de rand van de deelgraaf zijn toegewezen en voegen de kant toe met het kleinste getal. Het eindpunt wordt een nieuw punt in de puntenverzameling van de graaf. Deze procedure wordt oneindig vaak herhaald, wat resulteert in een stochastische deelgraaf van $\mathbb{Z}^2$.

In Hoofdstuk 3 bestuderen we eigenschappen van de verkregen deelgraaf, het zogenaamde veroverde gebied (invaded region), met de nadruk op bepaalde delen van dit gebied: de invasie meren. We leiden machtswetten af voor de verdeling van enkele karakteristieke grootheden met betrekking tot deze meren. We vergelijken ook het veroverde gebied met een andere stochastische graaf, het emergent oneindig cluster en we laten zien dat de modellen op lokaal niveau op elkaar lijken maar significant van elkaar verschillen op globaal niveau.

Het laatste model dat we in dit proefschrift beschouwen is een stochastische dynamiek voor het al eerder genoemde emergent oneindig cluster. Dit cluster is wederom een oneinige stochastische deelgraaf van $\mathbb{Z}^2$ dat via een bepaalde procedure verkregen wordt. Om de kansverdeling van dit cluster te kunnen bestuderen, presenteren we drie dynamische modellen in Hoofdstuk 4 als benadering voor de kansverdeling van het emergent oneindig cluster. Omdat de graaf oneindig is, en een lokale actie kan afhangen van de situatie op zeer grote afstand, is het niet direct duidelijk dat bovenstaande processen bestaan (d.w.z. wiskundig rigoreus geconstrueerd kunnen worden). In Hoofdstuk 4 tonen we het bestaan van de drie dynamische modellen aan door een wiskundig formele constructie te geven.